Precalculus

Volume 1

KJ Tim McDonald

Preamble: What is Precalculus?

Precalculus is the study of functions and their graphs. As its name indicates, it is undertaken in order to prepare the student for the study of Calculus itself.

Calculus has two main branches.

The first is differentiation. For any function, $y = f(x)$, this begins with finding the slope or gradient m of the tangent to the graph of $y = f(x)$ at any point $(a, f(a))$. See Figure 0.1.

The second is integration. This begins with the calculation of the area A between the graph of the function (a curve) and an interval $[a,b]$ on the $x-axis$. See Figure 0.2.

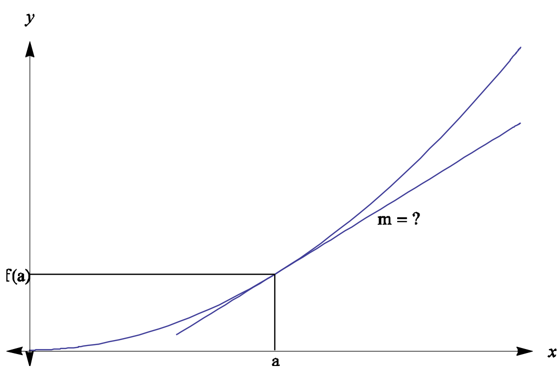


Figure 0.1 Differentiation

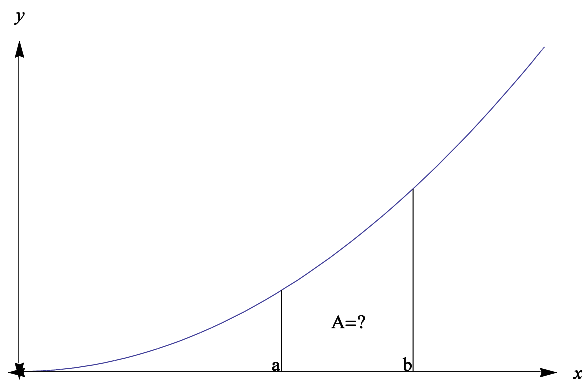


Figure 0.2 Integration

The techniques you will learn in Calculus apply to any function and its graph and in order to undertake Calculus, you will need to be able to sketch the graphs of many different functions.

From your algebra courses, you know the graphs of a number of basic functions such as,

$$f(x) = x, x^2, x^3, \sqrt{x}, \sqrt[3]{x}, |x|, \frac{1}{x}, \frac{1}{x^2}$$

as well as the graphs of the relations:

Circle $x^2 + y^2 = r^2$, Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and the other conic section, the parabola given by the function $x^2 = 4py$.

In Precalculus, you will extend your knowledge of these basic graphs and determine the equations and graphs of transformations due to translations, reflections, shrinking and stretching of these basic graphs. For example, you may recall from your algebra classes that the graph of $f(x) = x^2 - 2x + 1 = (x-1)^2$ has exactly the same shape as the basic graph $f(x)=x^2$ but is translated one unit to the right and the graph of $f(x)=-x^2$ also has exactly the same shape but is reflected in the $x-axis$ (see Figures 0.3, 0.4 and 0.5 below).

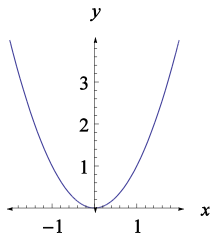


Figure 0.3 $y=x^2$

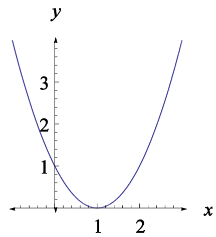


Figure 0.4 $y=(x-1)^2$

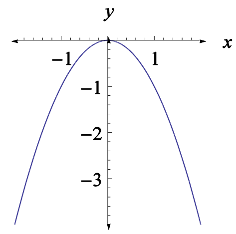


Figure 0.5 $y=-x^2$

When you come to learn Calculus where you are required to differentiate and integrate these extended functions, you should now be able to draw them with ease and be freed to concentrate on the Calculus questions without having to first spend an inordinate amount of time in drawing graphs!

In Precalculus, you will learn one way to define new functions is to form inverse functions from these basic ones and their extensions. We will see this is possible under certain conditions.

You will also be able to extend the repertoire of basic functions, their extensions and their inverses, to others that occur frequently in the development of mathematical models used in the study of very many branches of knowledge, not just the sciences such as physics, chemistry, biology, geology and neuroscience, but also in disciplines such as economics, demographics, insurance, ecology, medicine, sociology and psychology.

The first group of these is the exponential functions together with their inverse logarithmic functions. You may have encountered the natural exponential function, $y=e^x$, before. It occurs extremely frequently in mathematics and mathematical modeling. Each of these so-called transcendental functions has a basic graph. Each can be extended by translations, reflections, stretching and shrinking, in exactly the same manner as the basic algebraic functions.

The second is the family of trigonometric functions – the sine, cosine, tangent, cosecant, secant and cotangent functions – and their inverse functions. Again, these occur naturally in devising mathematical models to study a huge range of branches of knowledge. Each of these functions also has a basic graph. Each can be extended by translations, reflections, stretching and shrinking, in exactly the same manner as the basic algebraic functions.

You will learn to work with and manipulate these new functions and know how to graph them, so you will be prepared for Calculus and mathematical modeling in general, where they are used all the time!

Contents

Acknowledgements

Our thanks to the following faculty members and graduate students of the Department of Mathematics and Computer Science at Rutgers University, Newark, who reviewed all or parts of the manuscript. The time, effort and suggestions they contributed are very much appreciated.

Pritam Ghosh
Hetal Kapadia
William Keigher
David Keys
Zhengu Mao
Pallavi Parekh
Ravi Srivinasan

Note for instructors at Rutgers-Newark

At Rutgers-Newark, Chapters 1 and 2 of this text, with the exception of inverse functions discussed at the end of Chapter 2, have been covered in Math 109 College Algebra for the Physical Sciences. Accordingly, Chapters 1 and 2 will be dealt with expeditiously as revision.

In a typical Spring or Fall semester, allowing 5 classes for three reviews and two mid-term exams, the remaining 24 80-minute classes would be allocated approximately as follows:

Chapter 1:	2.5 classes
Chapter 2:	2.5 classes
Chapter 3:	3 classes
Chapter 4:	3 classes
Chapter 5:	3 classes
Chapter 6:	3 classes
Chapter 7:	3 classes
Chapter 8:	4 classes

Homework Assignments

When you purchase this book, you also purchase a code for access to WebAssign homework assignments. Quick Start instructions for accessing WebAssign are provided on the following page. In the first class you will be provided with the Class key. The homework assignments are mandatory and are a part of the final grade.

The exercises in the WebAssign assignments have been selected from a number of different textbooks. For each question you access it electronically from that particular book. In addition, you have free access to the full electronic book, Precalculus, 8^{th} edition, by Larson.

Note that WebAssign randomly selects different numbers each time it asks the same type of question. For example, Exercise 1 of Assignment 1 in this text is "Find the distance between the points $(-2,-1),(7,11)$." WebAssign might ask for the same exercise, "Find the distance between the points $(-3,-3),(5,3)$." If you answer incorrectly, then on your second attempt you might be asked: "Find the distance between the points $(-1,-1),(3,2)$", etc.

If you have difficulty using the palettes to insert symbols such as $\sqrt{\ }$, WebAssign suggest you switch to Google Chrome.

There is the option to use the Exercises in this book for homework and practice.

Notation

Elements of a set S
$a \in S$ is read "a is an element of the set S".

Set notation

- $\{x : condition\}$ is read " the set of x such that (the condition holds)"
- Union of two sets: If A and B are sets, $A \cup B$ (read "A union B") is the set containing the elements that are in either A or B or both, so $A \cup B = \{x : x \in A \ or \ x \in B\}$.
- Intersection of two sets: If A and B are sets, $A \cap B$ (read "A intersection B") is a set containing the elements in both A and B, so $A \cap B = \{x : x \in A \ and \ x \in B\}$.

Sets of numbers

$\mathbb{N}$ The set of the natural or counting numbers $\{1, 2, 3, ...\}$

$\mathbb{Z}$ The set of the integers $\{0, \pm 1, \pm 2, ...\}$

$\mathbb{Q}$ The set of the rational numbers $\left\{x = \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$, e.g. $0, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1 \in \mathbb{Q}$.

$\mathbb{R}$ The set of the real numbers, that is, numbers you can place on a real number line,

e.g., $0, \pm 1, \sqrt{2}, \sqrt[3]{11}, e, \frac{11}{3}, \pi \in \mathbb{R}$

$\mathbb{C}$ The set of complex numbers, $\{z = a + bi : a, b \in \mathbb{R}, \ i = \sqrt{-1}\}$

Logical connectors

$\Rightarrow$ $P \Rightarrow Q$ is read " if P then Q" or "P implies Q"

$\Leftrightarrow$ $P \Leftrightarrow Q$ is read " P and Q are equivalent (statements)",
or " if P then Q <u>and</u> if Q then P", or "P <u>if and only if</u> (iff) Q".

Interval notation
With reference to the $x-axis$ or a real number line labeled x, we have the following notation for intervals,

$$(a,b) = \{x : a < x < b\}$$
$$[a,b] = \{x : a \le x \le b\}$$
$$[a,b) = \{x : a \le x < b\}$$
$$(a,b] = \{x : a < x \le b\}$$
$$[a,\infty) = \{x : x \ge a\}$$
$$(a,\infty) = \{x : x > a\}$$
$$(-\infty,b) = \{x : x < b\}$$
$$(-\infty,b] = \{x : x \le b\}$$

Chapter 1

Cartesian Coordinate System

1.1 Cartesian or Rectangular Coordinate System

To identify locations or points on a plane, René Descartes (1596-1650) drew two number lines at right angles to one another, placing one horizontally, (the $x-axis$), and the other vertically, (the $y-axis$), so that the zeros on each axis corresponded at what we call the origin $(0,0)$.

The two axes divide the plane into four quadrants (Q), labeled counterclockwise from the top right as Q1, Q2, Q3 and Q4 as shown in Figure 1.1.

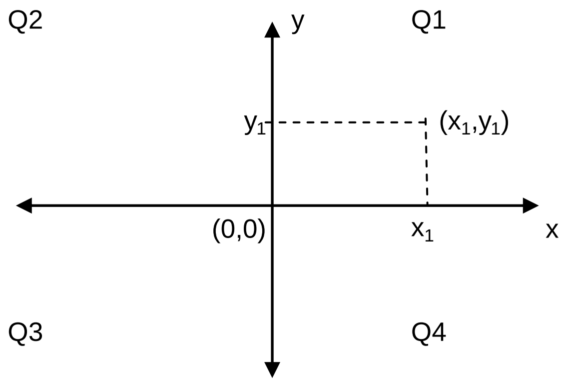


Figure 1.1

Any point on the plane then lies vertically opposite a number on the $x-axis$,which we call its $x-coordinate$, say x_1, and horizontally opposite a number on the $y-axis$, which we call its $y-coordinate$, say y_1. In deference to Descartes they are called Cartesian coordinates. If we draw vertical and horizontal lines from the point to the respective x and y axes, then we have a rectangle, hence the name rectangular coordinate system, showing the point (x_1, y_1). Note that points on the plane are labeled (x, y). Each has an *ordered* pair of coordinates, with the $x-coordinate$ before the $y-coordinate.$

1.2 Relations

A relation is a set of ordered pairs (x, y). The values x may take are called the domain, the values y may take are called the range. In algebra, we normally deal with relations specified by an equation between the two variables.

For example, $y = x+3$ is an equation specifying the relation between the two variables x and y. This equation is obeyed by the ordered pair $(3,6)$ since $6 = 3+3$ but not by $(2,4)$ since $4 \neq 2+3$. We say $(3,6)$ is a solution of the equation but $(2,4)$ is not.

In the case of $y = x + 3$ we call x the independent variable and y the dependent variable. That is, given any value of x, say $x = 4$, the value of y is fully determined, namely, $y = 4 + 3 = 7$.

1.3 Graphs of equations

If an equation gives a relationship between the variables x and y, for example $3x + 4y = 12$, then we can plot all the points on the coordinate plane which satisfy this relationship. This is called the graph of the equation. There are generally an infinite number of such points, including in this case the points $(4,0), (0,3), (8,-3), (-4,6)$. Here, we just need to identify any two, plot them and then use a ruler to draw (part of) the line.

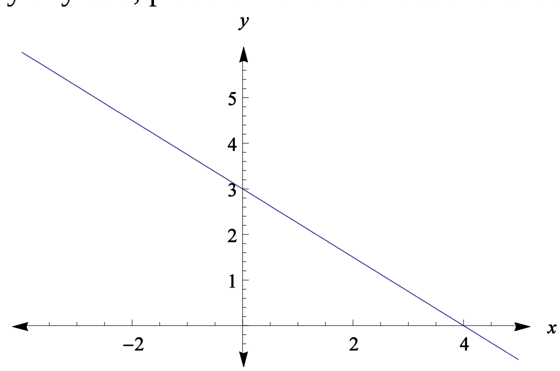


Figure 1.2 Graph of $3x + 4y = 12$

1.4 Distance formula

The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the coordinate plane is

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This follows immediately from the Pythagorean Theorem applied to Figure 1.3.

$$P_1P_2^{\,2} = P_1Q^2 + P_2Q^2$$

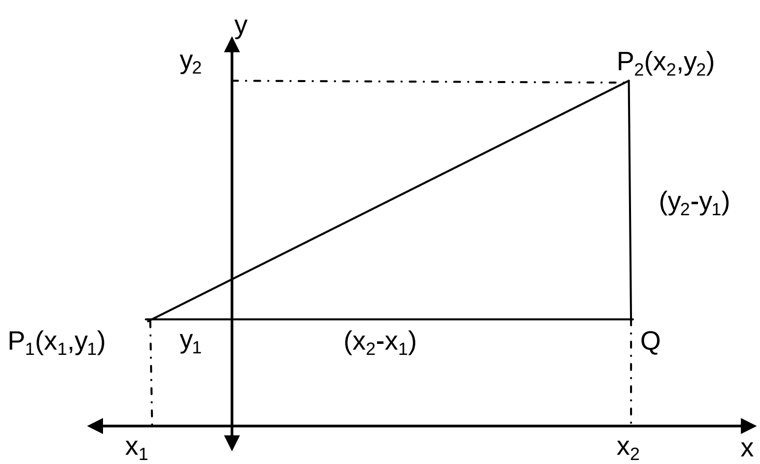


Figure 1.3

Example 1
Find the distance between the points A$(2,3)$ and B$(-1,5)$

<u>Solution:</u> $AB=\sqrt{(-1-2)^2+(5-3)^2}=\sqrt{13}$

1.5 Midpoint Formula

The midpoint of two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is the point $M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

Proof: Exercise.

Example 2
Find the midpoint of the line segment joining $(3,4)$ and $(-1,-8)$

<u>Solution</u>: $\left(\frac{3-1}{2},\frac{4-8}{2}\right)=(1,-2)$

1.6 Equation of a Circle

We define a circle to be the set of points on the coordinate plane that are equidistant from a given point, called the center.

1.6.1 Equation of a circle center $(0,0)$ and radius r

If the center is $(0,0)$, the common distance is r and (x,y) is any point on the circle, then by using the distance formula or the Pythagorean Theorem, we obtain $x^2+y^2=r^2$ as the equation of the circle (the relationship all points on the circle must satisfy).
See Figure 1.4.

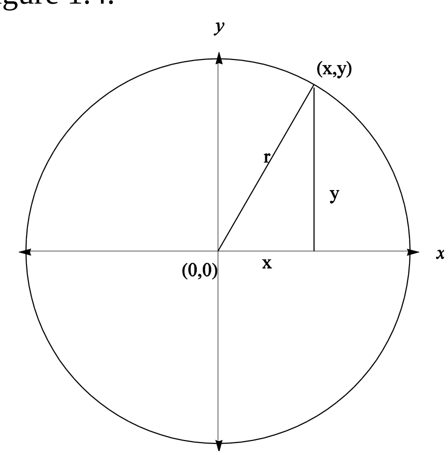


Figure 1.4 $x^2+y^2=r^2$

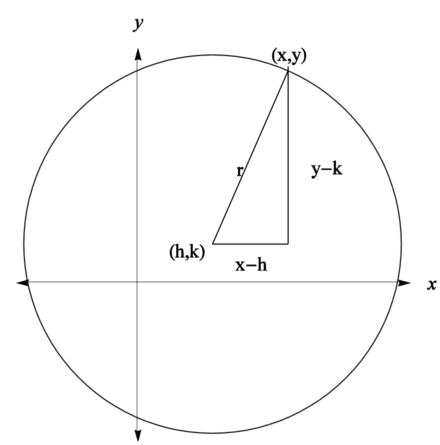


Figure 1.5 $(x-h)^2+(y-k)^2=r^2$

1.6.2 Equation of a circle center (h,k) and radius r

In the same way, if (x,y) is any point on the circle center (h,k) and radius r, then by using the distance formula or the Pythagorean Theorem, we obtain $(x-h)^2+(y-k)^2=r^2$ as the equation. See Figure 1.5.

Example 3
Find the equation of the circle center (0,0) of radius 3.
Solution: $x^2 + y^2 = 9$

Example 4
Find the equation of the circle center (-3,4) of radius 4.
Solution:

$$(x+3)^2 + (y-4)^2 = 16$$
$$\Leftrightarrow x^2 + y^2 + 6x - 8y + 9 = 0$$

Example 5
Sketch the graph of the equation $x^2 + y^2 - 4x + 6y - 3 = 0$
Solution:
We need to put the equation in the form $(x-h)^2 + (y-k)^2 = r^2$ so we know its center is (h,k) and its radius is r. We do this by completing the square[1] for both the x and y terms.

$$x^2 + y^2 - 4x + 6y - 3 = 0$$
$$\left(x^2 - 4x + 4\right) - 4 + \left(y^2 + 6y + 9\right) - 9 - 3 = 0$$
$$(x-2)^2 + (y+3)^2 = 16$$

So the center is $(2,-3)$ and the radius is 4. See Figure 1.6.

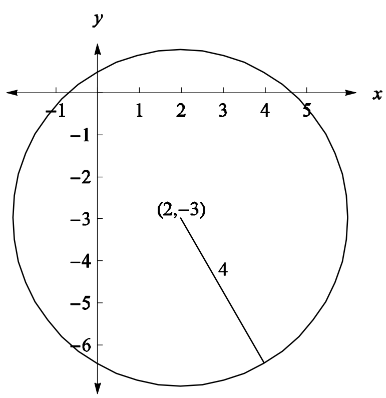


Figure 1.6 $x^2 + y^2 - 4x + 6y - 3 = 0$

---

[1] Completing the square for terms of the form $x^2 + 2bx$ is done by noting that a perfect square expands by $(x+c)^2 = x^2 + 2cx + c^2$, that is, its expansion is "square the first, plus twice the product, plus square the second". So we write $x^2 + 2 \bullet b\, x = x^2 + 2 \bullet b\, x + b^2 - b^2 = \left(x+b\right)^2 - b^2$. This is a perfect square minus "square the second". The method is, add and subtract the square of half the coefficient of x.
Example: $x^2 + 6x = x^2 + 2 \bullet 3\, x = x^2 + 2 \bullet 3\, x + 9 - 9 = (x+3)^2 - 9$

Example 6
Find the equation of a circle with the line segment joining the points A$(1,2)$ and B$(5,6)$ as a diameter.

Solution: We use the midpoint formula to find the center $C\left(\frac{1+5}{2},\frac{2+6}{2}\right)=C(3,4)$

Then we use the distance formula to find the radius AC or BC (or we can find AB and divide by 2).

$$r = \text{AC} = \sqrt{(3-1)^2+(4-2)^2} = \sqrt{8}$$

So the equation of the circle is $(x-3)^2+(y-4)^2=8$. See Figure 1.7a.

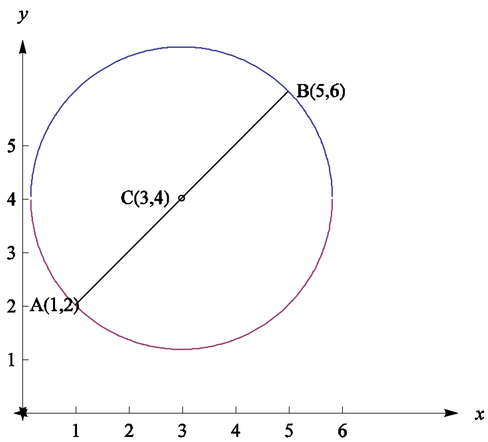


Figure 1.7a $(x-3)^2+(y-4)^2=8$

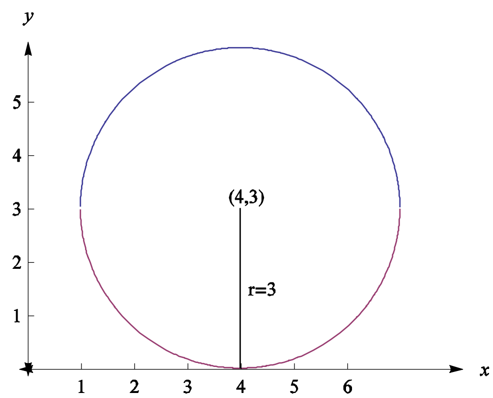


Figure 1.7b $(x-4)^2+(y-3)^2=9$

Example 7
What is the equation of a circle center (4,3) that is tangential to the x-axis?
Solution: It must have radius 3. Answer: $(x-4)^2+(y-3)^2=9$. See Figure 1.7b.

1.7 Straight Lines

A straight line is defined as the (infinite) set of points that satisfy a given linear equation of the form $ax+by+c=0$. Geometrically, we could define a straight line as the movement of a point in a constant direction. See Figure 1.8.

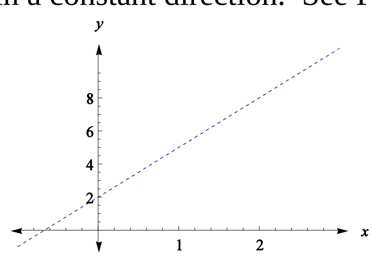


Figure 1.8

1.7.1 Slope of a line joining two points

We define the slope m of the line joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the coordinate plane by $m = \dfrac{change\ in\ y}{change\ in\ x} = \dfrac{\Delta y}{\Delta x}$. The two changes are measured as we move left to right. From Figure 1.9, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

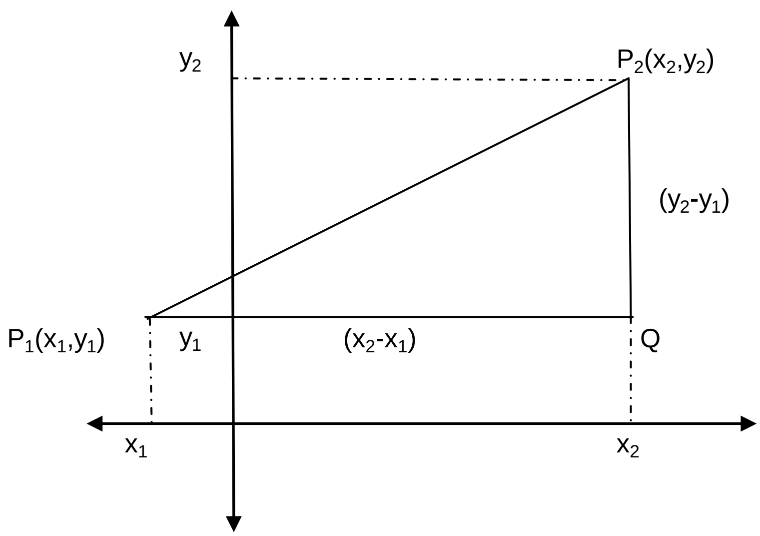


Figure 1.9

If we take $P_2(x_2, y_2)$ to be to the right of $P_1(x_1, y_1)$, then $x_2 - x_1$ will always be positive. If the line rises from left to right, then $y_2 - y_1$ will be positive also, but if the line falls from left to right then $y_2 - y_1$ will be negative. It follows that the slope (or gradient) of line will be positive if the line rises from left to right, but negative if the line falls from left to right. See Figure 1.10.

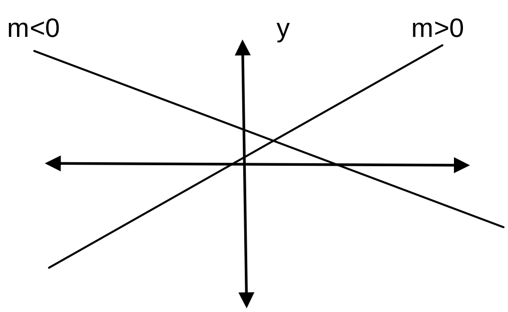


Figure 1.10

Example 8

Find the slope of the line joining the two points $(-1,-4)$ and $(3,4)$

Solution: $m=\dfrac{4-(-4)}{3-(-1)}=2$

Example 9

Find the slope of the line joining the two points $(-3,4)$ and $(1,-3)$

Solution: $m=\dfrac{-3-4}{1-(-3)}=-\dfrac{7}{4}$

1.7.2 Equation of a straight line in point-slope form

To find the equation of a line through a given point $P_1(x_1,y_1)$ of slope m, we identify $P(x,y)$ as any point on the line. Then (see Figure 1.11) the relationship between x and y is easily obtained from the slope formula, namely,

$$m=\frac{y-y_1}{x-x_1}\Rightarrow y-y_1=m(x-x_1)$$

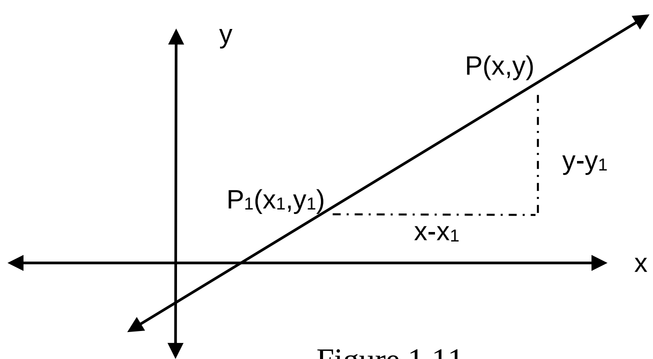


Figure 1.11

Example 10

Find the equation of the line with slope $\frac{1}{2}$ that passes through the point $(-2,1)$

Solution: $y-y_1=m(x-x_1)\Rightarrow(y-1)=\frac{1}{2}(x-(-2))\Leftrightarrow x-2y+4=0$

Note: We always put the equation of a line in the standard form $ax+by+c=0$ for $c>0$. If $c<0$, we may write it as $ax+by=-c$. For example, we can write $3x-y-6=0$ or $3x-y=6$.

Example 11
Find the equation of the line joining the two points $(1,3)$ and $(-3,-5)$

Solution: We first find the slope $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{-5-3}{-3-1}=2$

Then the equation comes from $y-y_1=m(x-x_1)\Rightarrow y-3=2(x-1)\Leftrightarrow 2x-y+1=0$

1.7.3 Equation of a straight line in slope- $y-$intercept form

In the particular case where the given point is on the $y-axis$, say $P_1(0,b)$, the equation becomes $\quad y-y_1=m(x-x_1)\Rightarrow y-b=mx\Leftrightarrow y=mx+b$
We call $P_1(0,b)$ the $y-$intercept. It is the point where the line crosses the $y-axis$. See Figure 1.12.

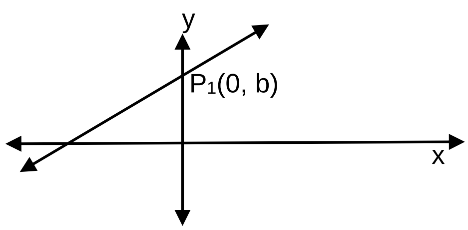


Figure 1.12

Example 12
Find the equation of the line with slope -2 and y-intercept -5.
Solution: $y=mx+b\Rightarrow y=-2x-5\Leftrightarrow 2x+y+5=0$

Example 13
Find the slope of the line with equation $3x-7y=11$

Solution: We write $3x-7y=11\Leftrightarrow y=\dfrac{3}{7}x-\dfrac{11}{7}$ so it is in the form $y=mx+b$. Then its slope is $\dfrac{3}{7}$ (and its $y-$intercept is $\dfrac{-11}{7}$)

Example 14
Sketch the graph of the line with slope $\dfrac{3}{2}$ passing through $(1,1)$ and label another point on it by using the value of its gradient only.

Solution: Since the gradient is $m=\dfrac{change\,in\,y}{change\,in\,x}=\dfrac{3}{2}$, starting at $(1,1)$ we increase x by 2 to reach $(3,1)$ and then y by 3 to arrive at the point $(3,4)$. Then just draw the line containing these two points. See Figure 1.13.

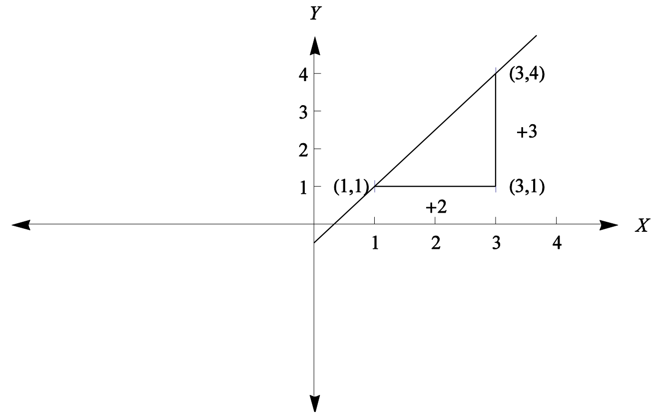


Figure 1.13

1.7.4 Sketching the graph of the equation of a straight line

Since the equation of a straight line with slope m is found by using $y - y_1 = m(x - x_1)$, it follows that all straight lines have equations of the form $ax + by + c = 0$ where a, b, c are real numbers. To graph a line we just need to plot two points and then put a ruler along them. The easiest two points to find are the intercepts on the axes. The $x-$intercept has the form $(x, 0)$ and the $y-$intercept has the form $(0, y)$, so we just need to complete the table

x	y
?	0
0	?

Example 15
Sketch the line with equation $3x - 4y = 12$
<u>Solution:</u>

x	y
$3x = 12$	0
0	$-4y = 12$

$\Rightarrow$

x	y
4	0
0	-3

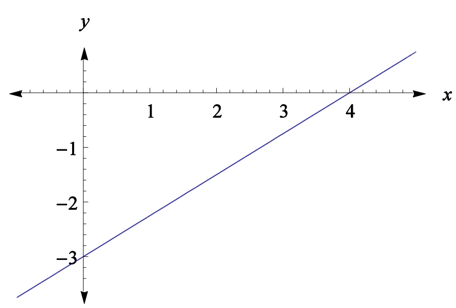


Figure 1.14 $3x - 4y = 12$

1.7.5 Graphs of $x = a,\ y = b$

The graph of $x = a$ is a straight line that passes through all the points with an $x-$coordinate of $a.$ The $y-$coordinates can be any numbers at all. Therefore the graph of $x = a$ is a vertical line paralleling the $y-axis.$ Note its slope is undefined.

Similarly, the graph of $y = b$ is a straight line that passes through all the points with a $y-$coordinate of $b.$ The $x-$coordinates can be any numbers at all. Therefore the graph of $y = b$ is a horizontal line paralleling the $x-axis.$ Note its slope is zero. An example is given in Figure 1.14.

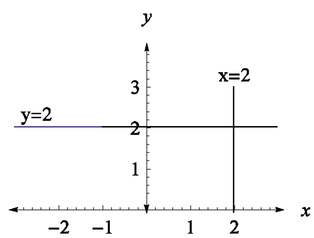


Figure 1.14 Graphs of $x = 2, y = 2$

Example 16
Sketch the figure bounded by $x = 2,\ x = -2,\ y = 0,\ y = 6$ and find its area.
Solution: The two x lines are 4 units apart, and the two y lines are 6 units apart. The bounded figure is a 6×4 rectangle of area 24. See Figure 1.15.

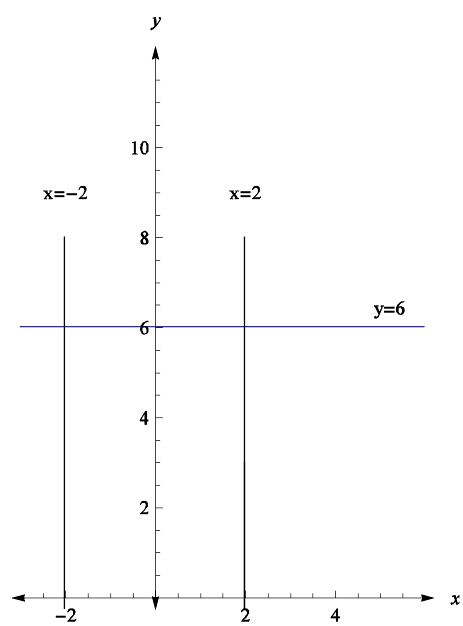


Figure 1.15 $x = 2,\ x = -2,\ y = 0,\ y = 6$

1.7.6 Parallel lines.

Two parallel lines must have the same slope. See Figure 1.16.

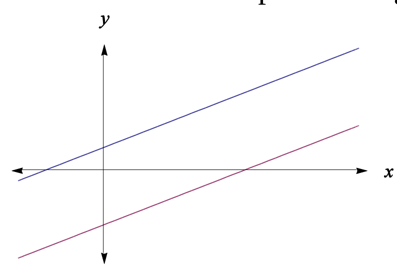


Figure 1.16 Parallel Lines

The slope of $ax+by+c=0 \Leftrightarrow y=-\frac{a}{b}x-\frac{c}{a}$ is $-\frac{a}{b}$ (by comparison with $y=mx+b$).

Thus the slope of any parallel line must also be $-\frac{a}{b}$.

Example 17

Find the equation of the line passing through $(-1,2)$ that is parallel to the line with equation $7x-5y+13=0$.

Solution: Since $7x-5y+13=0 \Leftrightarrow y=\frac{7}{5}x+\frac{13}{5}$, the two lines must have slope $\frac{7}{5}$.

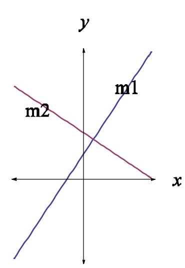
Use $y-y_1=m(x-x_1) \Rightarrow y-2=\frac{7}{5}(x+1) \Leftrightarrow 7x-5y+17=0$

1.7.7 Perpendicular lines

We can prove that the product of the slopes m_1 and m_2 of two perpendicular lines is $m_1m_2=-1 \Leftrightarrow m_2=-\frac{1}{m_1}$. That is, the second slope is the negative reciprocal of the first.

This is proved by shifting the lines so that they intersect at the origin (second figure of Figure 1.17) and then using the Pythagorean theorem on the triangle OAB.

You are encouraged to complete the details.



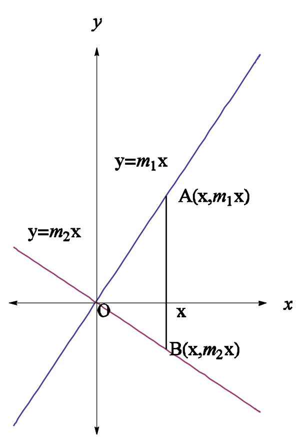


Figure 1.17 Perpendicular Lines

Example 18

Find the equation of the line passing through $(-1,2)$ that is perpendicular to the line with equation $7x-5y+13=0.$

<u>Solution:</u> Since the slope of the first line is $m_1=\frac{7}{5}$, the slope of the second line is $m_2=-\frac{5}{7}$. Use $y-y_1=m(x-x_1)\Rightarrow y-2=-\frac{5}{7}(x+1)\Leftrightarrow 5x+7y-9=0$

Exercises 1A

1. Find the distance between the points $(-3,-5),(6,7)$
2. Find the distance between the points $(3,-4),(6,-7)$
3. Find the center and radius of the circle $(x+2)^2+(y-5)^2=36$
4. Find the center and radius of the circle $x^2+y^2-6x-8y+9=0$
5. Find the equation of the circle that passes through the origin and has its center at $(-3,-4)$
6. Find the equation of the circle for which the line segment determined by $(-4,9)$ and $(10,-3)$ is a diameter.
7. Find the slope of the line containing the two points $(1,2),(4,6)$.
8. Find x if the line through $(x,5)$ and $(1,2)$ has a slope of $\frac{3}{5}$
9. Graph the line that passes through $(0,5)$ and has slope $m=-\frac{1}{4}$
10. Find the equation of the line with slope $m=3$ containing the point $(-2,4)$
11. Find the equation of the line containing the points $(2,1),(6,5)$
12. Find the equation of the line containing the points $(0,0),(-5,9)$
13. Write the equation of the line with slope $m=2/9$ and $y-$intercept $b=3.$
14. Write the equation of the line with $x-$intercept -3 and slope $-\frac{5}{8}$
15. Change the equation $3x+y=7$ to slope and $y-$intercept form and find the slope and $y-$intercept
16. Find the slope and $y-$intercept of $x+y=3$ and draw its graph.
17. Find the slope and $y-$intercept of $x+3y=0$ and draw its graph.
18. Find the slope and $y-$intercept of $y=4$ and draw its graph.
19. Find the slope and $y-$intercept of $x=-5$ and draw its graph.
20. Write the slope-intercept forms of the equations of the lines through $(-3,2)$ (a) parallel, and (b) perpendicular to the line $x+y=7$
21. Write the slope-intercept forms of the equations of the lines through $(7/8,3/4)$ (a) parallel, and (b) perpendicular to the line $5x+3y=0$

1.8 Graphing polynomial functions in one independent variable.

Definition

A polynomial function of degree n **in the independent variable** x has the general form

$$y = f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

where n is a whole number, $n \in \{0,1,2,3,...\}$, and the a_i 's are real numbers.

1.8.1 Graphing quadratic functions

A quadratic function is a polynomial of degree 2. The simplest (the one we will call the basic) is $y = x^2$. See Figure 1.18.

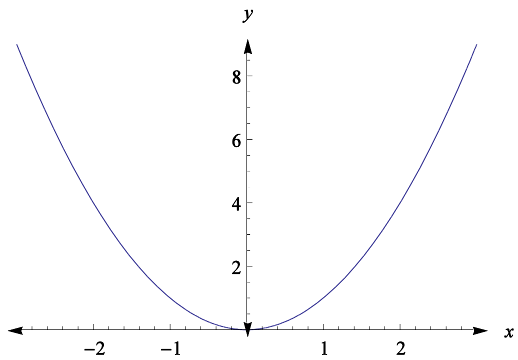


Figure 1.18 $y = x^2$

The graphs of all other polynomials of the form $y = ax^2 + bx + c$ have the same general shape, called a parabola, as $y = x^2$. They are easily sketched by finding their intercepts on the axes, given by

x	y
$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	0
0	c

Note there may be 0, 1 or 2 $x-$intercepts according to whether $b^2 - 4ac$ is negative, zero or positive respectively. See Figures 1.19, 1.20 and 1.21.

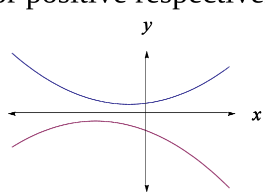


Figure 1.19 $b^2 - 4ac < 0$
No $x-$intercept

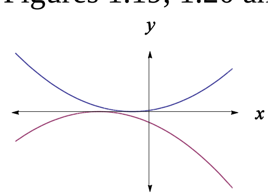


Figure 1.20 $b^2 - 4ac = 0$
One $x-$intercept

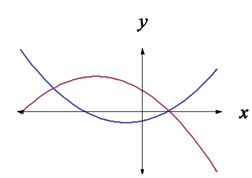


Figure 1.21 $b^2 - 4ac > 0$
Two $x-$intercepts

The vertex of a parabola may be determined by noting it is symmetrical about a line through its vertex (maximum or minimum) point. So if x_1 and x_2 are the $x-axis$ intercepts, then the $x-$coordinate of the vertex is given by their midpoint $\frac{x_1+x_2}{2}$. The $y-$coordinate may then be found by substitution.

Example 19

Sketch the graph of $y=x^2-2x-3$

Solution:

Intercepts:

Factor:

$y=x^2-2x-3$

$=(x-3)(x+1)$

x	y
-1,3	0
0	-3

Vertex:

$x=\frac{3-1}{2}=1$

$y=f(1)=1-2-3=-4$

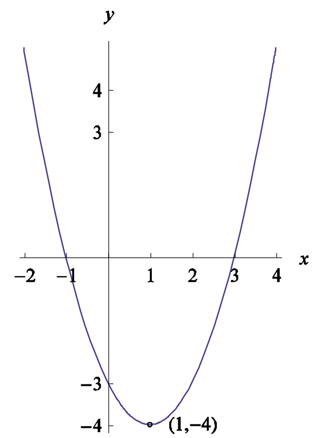


Figure 1.22 $y=x^2-2x-3$

Example 20

Sketch the graphs of $y=x^2-2x+1$, $y=x^2-4x-6$, $y=x^2+4x+8$

Solutions:

x	y
1,1	0
0	1

x	y
$(4\pm\sqrt{40})/2$	0
0	-6

x	y
$(-4\pm\sqrt{-16})/2$	0
0	8

Vertex: (1,0)

$$x=\frac{\frac{4+\sqrt{40}}{2}+\frac{4-\sqrt{40}}{2}}{2}=2$$

$$y=f(2)=4-8-6=-10$$

$$x=\frac{\frac{-4+\sqrt{-16}}{2}+\frac{-4-\sqrt{-16}}{2}}{2}=-2$$

$$y=f(-2)=(-2)^2-8+8=4$$

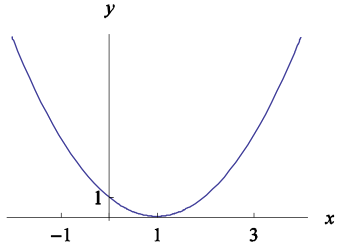


Figure 1.23 $y=x^2-2x+1$

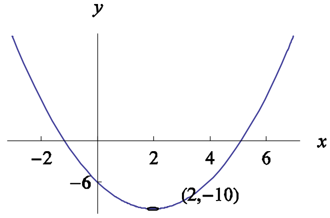


Figure 1.24 $y=x^2-4x-6$

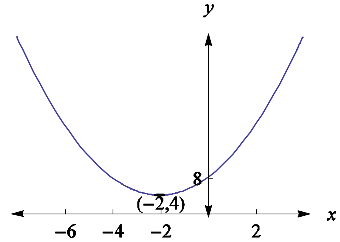


Figure 1.25 $y=x^2+4x+8$

1.8.2 Graphing quadratic inequalities

The solution of an inequality such as $ax^2+bx+c\geq 0$ is the set of $x-$values for which $y\geq 0$ where $y=ax^2+bx+c$. The answer is most easily obtained from the graph of $y=ax^2+bx+c$. (There are four inequality signs, $>,\geq,<,\leq$. All are treated by the same method).

Example 21

Solve $x^2-2x-3>0$

<u>Solution:</u> The solution to $x^2-2x-3>0$ is the set of $x-$values for which $y>0$ where $y=x^2-2x-3$. We simply plot the graph of $y=x^2-2x-3=(x-3)(x+1)$ and identify the values of x for which the graph has positive values of y. Equivalently, the graph is in the first or second quadrants so it is "above" the $x-axis$.

x	y
0	-3
$-1,3$	0

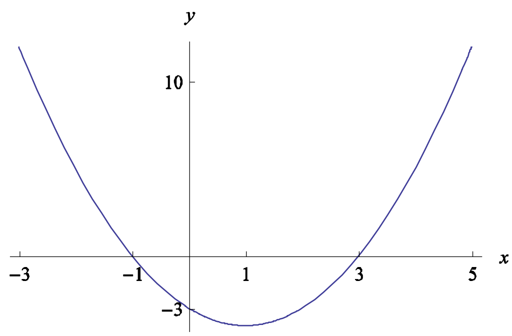


Figure 1.26 $y=x^2-2x-3$

The solution to $y>0$ is $\{x:x<-1 \text{ or } x>3\}\Leftrightarrow(-\infty,-1)\cup(3,\infty)$

Example 22

Solve $x^2-6x-3\geq 0$

<u>Solution:</u> To sketch $y=x^2-6x-3$, we use the table of values and Figure 1.27,

x	y
0	-3
$3\pm 2\sqrt{3}$	0

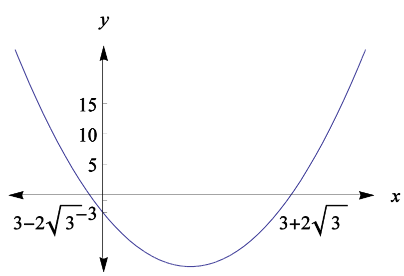


Figure 1.27 $y=x^2-6x-3$

Answer: $\{x:x\leq 3-2\sqrt{3} \text{ or } x\geq 3+2\sqrt{3}\}\Leftrightarrow(-\infty,3-2\sqrt{3}]\cup[3+2\sqrt{3},\infty)$.

Example 23

Solve $-x^2+3x-5>0$

<u>Solution:</u> To sketch $y=-x^2+3x-5$, we use the table of values and Figure 1.28,

x	y
0	-5
$\dfrac{-3\pm\sqrt{-11}}{-2}$	0

Answer:
No solutions. The graph does not intersect the $x-axis$. It is never "above" the $x-axis$.

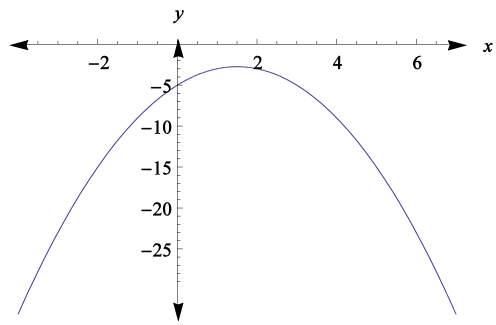


Figure 1.28 $y=-x^2+3x-5$

Example 24

Solve $x^2-4x+4\geq 0$

<u>Solution:</u> To sketch $y=x^2-4x+4=(x-2)^2$, we use the table of values and Figure 1.29,

x	y
0	4
2, 2	0

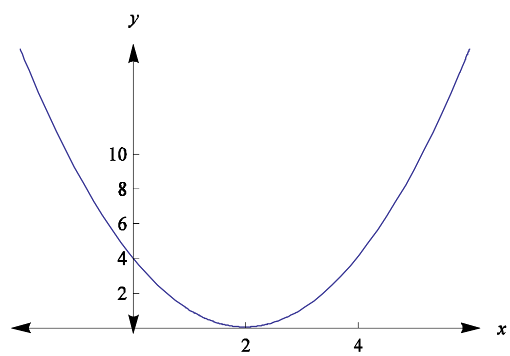


Figure 1.29 $y=x^2-4x+4$

Answer: $\{x:-\infty<x<\infty\}$ or $\mathbb{R}$ or $(-\infty,+\infty)$

1.8.3 Solving rational inequalities

We can use the same methods to solve simple rational inequalities.

Example 25

Solve $\dfrac{x+1}{x-5}>0$

<u>Solution</u>

First note the solution set cannot contain $x=5$.

$$\frac{x+1}{x-5}>0 \Leftrightarrow \frac{x+1}{x-5}\bullet\frac{x-5}{x-5}>0 \Leftrightarrow (x+1)(x-5)>0 \text{ since } (x-5)^2>0.$$

From Figure 1.30, the solution is
$\{x: x<-1 \ \ or \ \ x>5\}$
$\Leftrightarrow (-\infty,1)\cup(5,\infty)$

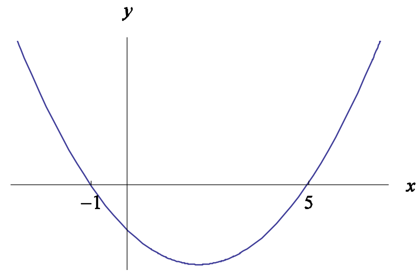


Figure 1.30 $y=(x+1((x-5)$

Example 26

Solve $\dfrac{2x-7}{x+5}\geq 3$

<u>Solution</u>
First note the solution set cannot contain $x=-5$.

$$\frac{2x-7}{x+5}\geq 3 \Leftrightarrow \frac{2x-7}{x+5}-3\geq 0 \Leftrightarrow \frac{2x-7-3x-15}{x+5}\geq 0$$

$$\Leftrightarrow \frac{-x-22}{x+5}\geq 0 \Leftrightarrow \frac{-x-22}{x+5}\bullet\frac{x+5}{x+5}\geq 0 \Leftrightarrow (-x-22)(x+5)\geq 0$$

Use Figure 1.31. The solution is NOT $\{x: -22\leq x\leq -5\}$ since $x=-5$ must be excluded.

The solution is therefore
$\{x: -22\leq x<-5\}\Leftrightarrow[-22,-5)$

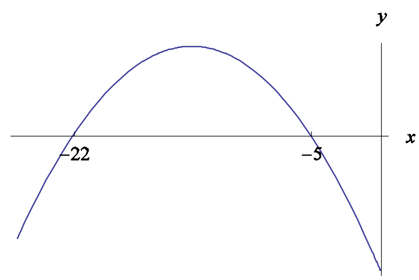


Figure 1.31 $y=(-x-22)(x+5)$

1.8.4 Graphing polynomials of degree ≥ 3

If the polynomial has factors, then we use the same technique of finding the intercepts on the axes. If one of the intercepts is at the origin, plot another point.

Example 27

Plot the graph of $y=x(x-2)(x+2)$.

<u>Solution</u> The graph is continuous. Plot the $x-$intercepts $0,-2,2$ but also the point $(3,15)$. See Figure 1.32.

x	y
0	0
$0, \pm 2$	0
3	15

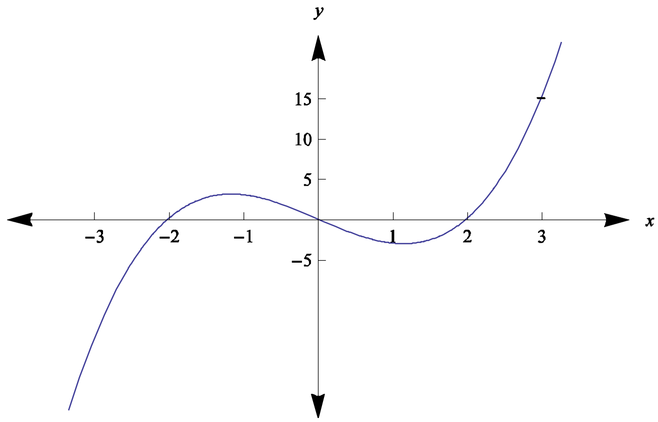


Figure 1.32 $y = x(x-2)(x+2)$

Example 28
Graph $f(x) = (x-1)(x+2)(x+1)$
<u>Solution:</u>

x	y
0	-2
$-1, -2, 1$	0
2	12

We also plotted the point $(2,12)$ in Figure 1.33.

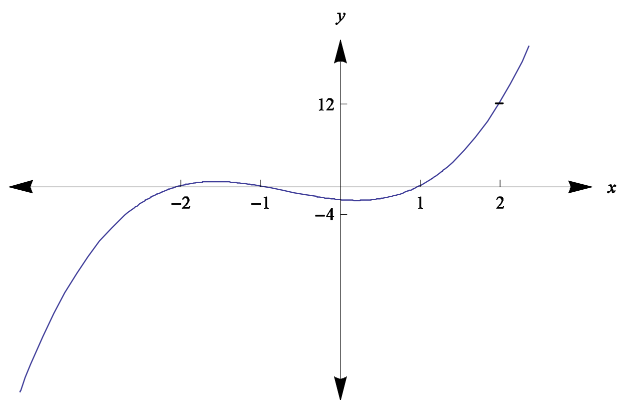


Figure 1.33 $f(x) = (x-1)(x+2)(x+1)$

1.8.5 Graphing inequalities of degree ≥ 3

Example 29
Solve $(x-1)(x+2)(x+1) \leq 0$
<u>Solution:</u> We use the same method as for quadratic inequalities. Sketch $f(x) = (x-1)(x+2)(x+1)$ and read off the answers. We can use Figure 1.33.
Answer: $\{x : x \leq -2 \ or \ -1 \leq x \leq 1\} \Leftrightarrow (-\infty, -2] \cup [-1, 1]$

Exercises 1B

1. Find the vertex and $x-$ and $y-$intercepts and sketch the graph of $f(x) = x^2 - 6x$
2. Find the vertex and $x-$ and $y-$intercepts and sketch the graph of $f(x) = x^2 + 4x + 3$
3. Sketch the graph of $f(x) = -x^2 - 2x + 1$
4. Sketch the graph of $f(x) = 2x^2 - 2x + 3$
5. Solve $(x+2)(x-1) > 0$
6. Solve $(2x-1)(3x+7) \geq 0$
7. Solve $x(x+2)(x-4) \leq 0$
8. Solve $x^2 + 2x - 35 < 0$
9. Solve $4 - x^2 < 0$
10. Solve $\dfrac{x+1}{x-2} > 0$
11. Solve $\dfrac{-x+2}{x-1} \leq 0$
12. Solve $\dfrac{2x}{x+3} > 4$
13. Solve $\dfrac{x-1}{x-5} \leq 2$
14. Sketch the graph of $y = x(x-1)^2$
15. Sketch the graph of $y = (x+2)(x-2)(x-3)$
16. Sketch the related graph and solve for $x : (x-2)(x-1)(x+4) \geq 0$
17. Sketch the related graph and solve for $x : (3-x)(1-x)(2+x) \leq 0$

Chapter 2

Sketching graphs of functions and relations

2.1 Relations

As stated in Section 1.2, a relation is a set of ordered pairs of numbers (x, y). The set of $x-values$ is called the domain and the set of $y-values$ is called the range. We can, of course, use any other symbols for x and y.

In algebra, we normally deal with relations specified by an equation between the two variables x and y. For example, $y = x+3$ is an equation specifying the relation between the two variables x and y. This equation is obeyed by the ordered pair $(3,6)$ since $6 = 3+3$ but not by $(2,4)$ since $4 \neq 2+3$. We say $(3,6)$ is a solution of the equation but $(2,4)$ is not.

In the case of $y = x+3$ we call x the independent variable and y the dependent variable. That is, given any value of x, say $x = 4$, the value of y is fully determined, namely, $y = 4+3 = 7$.

2.2 Functions

A function is a relation which assigns each element x in the domain to exactly one element y in the range. This is clearly the case with $y = x+3$ so we use the notation $y = f(x)$, spoken as "y is a function of x", where the particular function is $f(x) = x+3$.

If any value of the domain is assigned to more than one value of the range, as in the case $x = y^2$ (satisfied for example by both $(9,+3)$ and $(9,-3)$), then the relation is not a function. It follows that all vertical lines in the domain of a function can intersect the graph of the function at most once. This gives us the:

Vertical Line Test for a Function
A set of points in the coordinate plane is the graph of a function if and only if no vertical line intersects the graph at more than one point.

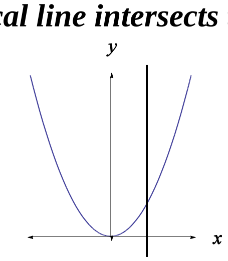


Figure 2.1 Function

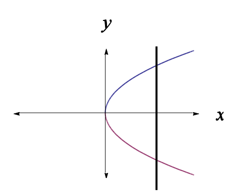


Figure 2.2 Not a function

You can immediately check that the relations $y = x^n$ are functions for all n, the relations $y^n = x$ are functions for odd n only and that the circle and ellipse are not the graphs of functions. For example, $y = x^2$ is the equation of a function (Figure 2.3), $x = y^2$ is not the equation of a function (Figure 2.4), the circle $x^2 + y^2 = 9$ is not the equation of a function (Figure 2.5), neither is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (Figure 2.6). See also Figures 2.7 to 2.12 below. They are all graphs of functions.

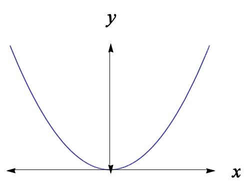


Figure 2.3 $y = x^2$

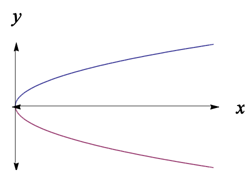


Figure 2.4 $x = y^2$

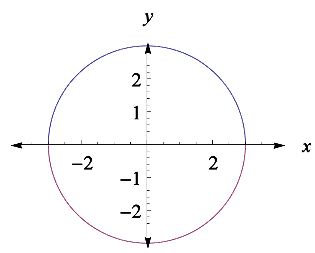


Figure 2.5 Circle

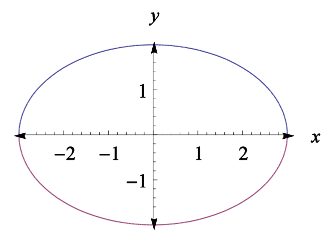


Figure 2.6 Ellipse

2.3 Graphs

A graph of a relation is a picture on a coordinate plane formed by the x and y axes which shows all the solutions of the equation for that relation. Of course, the allowable x values or domain of this set is often $(-\infty, +\infty)$, so we simply draw a subset of the solutions, generally those in the vicinity of the origin $(0,0)$.

2.3.1 Basic Graphs of Polynomial Functions

Definition
A **polynomial function of degree** n is a function of the form
$y = P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, $a_n \neq 0$, $a_i \in \mathbb{R}$, $n \in \mathbb{Z}, n \geq 0.$

The common graphs you have encountered in your algebra courses include the following <u>basic</u> shapes for polynomial functions.

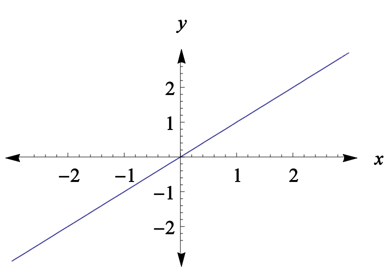


Figure 2.7 $y = x$

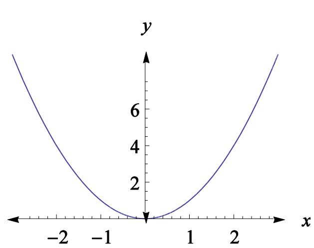


Figure 2.8 $y = x^2$

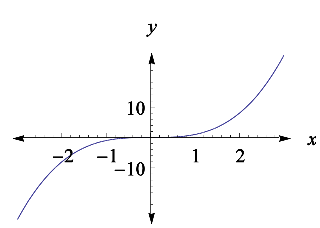


Figure 2.9 $y = x^3$

For $y = x^n$, $n > 0$, $n \in \mathbb{Z}$, obviously all these graphs pass through both the origin $(0,0)$ and the point $(1,1)$. Depending on whether n is odd or even, they also pass through $(-1,-1)$ or $(-1,1)$ respectively. As n increases, the section of the graph between $(0,0)$ and these two points becomes flatter and flatter.

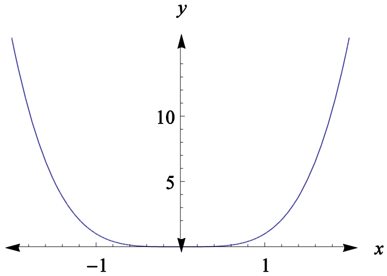


Figure 2.10 $y = x^4$

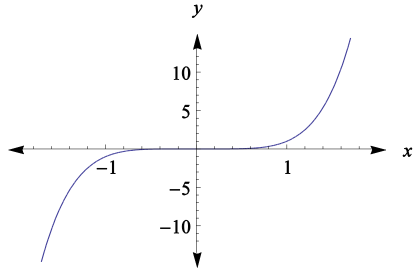


Figure 2.11 $y = x^5$

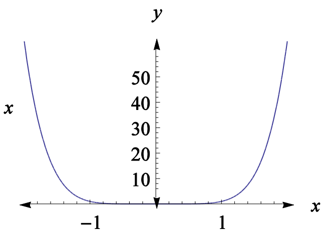


Figure 2.12 $y = x^6$

We note the graphs have two branches, one in the first quadrant, the other in the second or third quadrant depending on whether n is even or odd respectively.

2.3.2 Basic Graphs of Rational Functions

Definition

A **rational function** has the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Example: $f(x) = \dfrac{x-1}{x^2-4}$, $x \neq \pm 2$.

2.3.2.1 Basic Forms The basic forms are $y = x^n$, $n < 0$, that is, $y = \dfrac{1}{x}, y = \dfrac{1}{x^2}$, etc.

2.3.2.2 Graph of $y = \dfrac{1}{x}$, $x \neq 0$.

Although x cannot equal zero, let us study values of y as x approaches zero, written $x \to 0$. x can approach zero from the positive side (we say x approaches zero from above and write $x \to 0^+$) or from the negative side (we say x approaches zero from below and write $x \to 0^-$). Sample values are:

x	y
1	1
0.5	2
0.01	100
0.000001	1000000

x	y
-1	-1
-0.5	-2
-0.01	-100
-0.000001	-1000000

We conclude,

$$x \to 0^+ \Rightarrow y \to +\infty$$

$$x \to 0^- \Rightarrow y \to -\infty$$

The graphs of the two portions will be as shown in Figure 2.13.

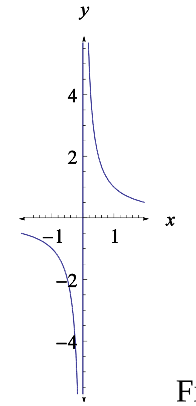


Figure 2.13

We see that as $x \to 0^+$, the graph never touches the positive $y-axis$ since $x \neq 0$, but its respective $y-coordinates$ rapidly become very large and positive. As $x \to 0^-$, the graph never touches the negative $y-axis$ since $x \neq 0$, but its respective $y-coordinates$ rapidly become very large and negative. We say the $y-axis$ or $x=0$ is a vertical asymptote.

Since $y=\frac{1}{x},\ x \neq 0 \Leftrightarrow x=\frac{1}{y},\ y \neq 0$, we could simply say the same shape occurs along the extremities of the $x-axis$. But, let us study $x \to \infty,\ x \to -\infty$. Sample values are,

x	y
1	1
2	0.5
100	0.01
1000000	0.000001

x	y
-1	-1
-2	-0.5
-100	-0.01
-1000000	-0.000001

We conclude,

$$x \to +\infty \Rightarrow y \to 0^+$$

$$x \to -\infty \Rightarrow y \to 0^-$$

The graphs of these two portions will be as shown in Figures 2.14 & 2.15

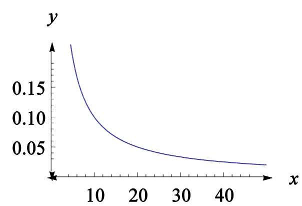


Figure 2.14

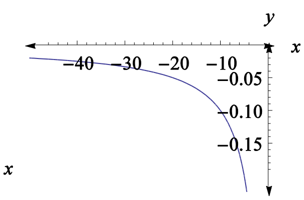


Figure 2.15

We say the $x-axis$ or line $y=0$ is a horizontal asymptote.

The full graph of $y=\frac{1}{x}$ is shown in 2.16.

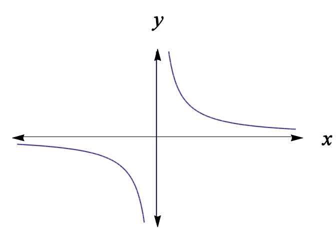


Figure 2.16 $y=\frac{1}{x}$

2.3.2.3 Graph of $y=\dfrac{1}{x^2}$, $x\neq 0$.

All values for y on this graph will be positive numbers. We have,

$x\to 0^+ \Rightarrow y\to +\infty$

$x\to 0^- \Rightarrow y\to +\infty$

The $y-axis$ or line $x=0$ is a vertical asymptote.

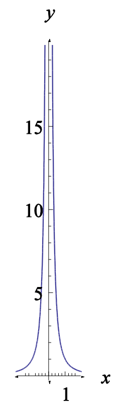


Figure 2.17

$x\to +\infty \Rightarrow y\to 0^+$

$x\to -\infty \Rightarrow y\to 0^+$

The $x-axis$ or line $y=0$ is a horizontal asymptote.

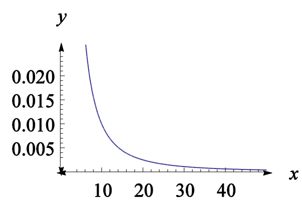


Figure 2.18

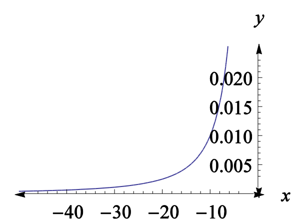


Figure 2.19

The graph is shown in Figure 2.20.

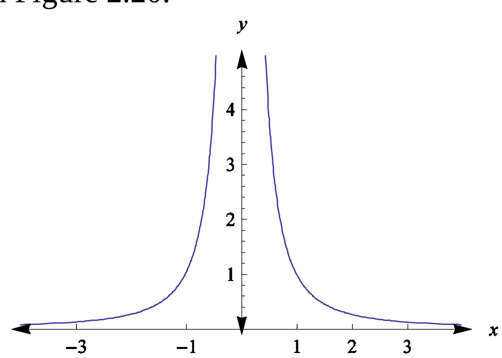


Figure 2.20 $y=\dfrac{1}{x^2}$

2.3.2.4 Graph of $y = \frac{1}{x^3}$, $x \neq 0$.

Since $x > 0 \Rightarrow y > 0$ and $x < 0 \Rightarrow y < 0$, the graph is similar to that of $y = \frac{1}{x}$.

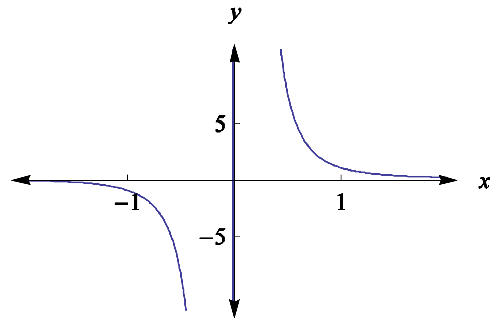


Figure 2.21 $y = \frac{1}{x^3}$, $x \neq 0$.

2.3.2.5 Summary

For $y = x^n$, $n < 0$, the graphs do not pass through $(0,0)$ or the axes since neither x nor y can equal 0. The graphs are all hyperbolas with two branches, one in the first quadrant, the other in the second or third quadrant depending on whether n is even or odd respectively. They all have $(1,1)$ as a solution, and depending on whether n is even or odd, they have $(-1,1)$ or $(-1,-1)$ respectively as another solution. We say the x and y axes are asymptotes – the graph cannot intersect either. However, as $x \to \pm\infty$ the graph progressively gets closer and closer to the $x-$axis (while never touching it) and as $x \to 0^{\pm}$ the graph progressively gets closer and closer to the $y-$axis (while never touching it).

We can use the procedure to sketch the graphs of other simple rational functions.

Example 1

Sketch the graph of $y = \frac{1}{x-1}$, $x \neq 1$.

<u>Solution</u>

The vertical asymptote is the line $x = 1$.

We argue,

$$x \to 1^+ \Rightarrow y \to +\infty$$

$$x \to 1^- \Rightarrow y \to -\infty$$

$$x \to +\infty \Rightarrow y \to 0^+$$

$$x \to -\infty \Rightarrow y \to 0^-$$

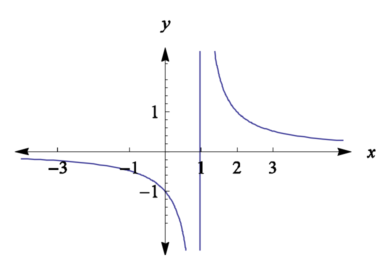


Figure 2.22 $y = \frac{1}{x-1}$, $x \neq 1$

Note: This is the graph of $y=\dfrac{1}{x}$ translated 1 unit to the right. In particular, the vertical asymptote moved 1 unit to the right. The horizontal asymptote remained unchanged. We will pursue this thought later.

Example 2

Graph $y=\dfrac{2x+1}{x} \Leftrightarrow y=\dfrac{1}{x}+2,\ x\neq 0$ (Note: divide to get the standard form)

Solution We argue,

$$x\to 0^{+} \Rightarrow y\to +\infty$$
$$x\to 0^{-} \Rightarrow y\to -\infty$$
$$x\to +\infty \Rightarrow y\to 2^{+}$$
$$x\to -\infty \Rightarrow y\to 2^{-}$$

The horizontal asymptote is now the line $y=2.$

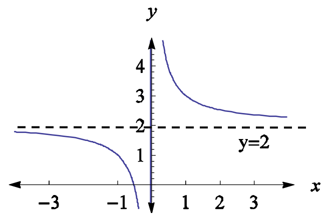


Figure 2.23 $y=\dfrac{1}{x}+2,\ x\neq 0$

Note: This is the graph of $y=\dfrac{1}{x}$ translated 2 units vertically up. In particular, the horizontal asymptote moved 2 units upwards. The vertical asymptote remained unchanged. We will also pursue this thought later.

Example 3

Graph $y=\dfrac{1}{(x-1)(x+1)},\ x\neq \pm 1.$

Solution

There are two vertical asymptotes, $x=1, x=-1.$

We argue and sketch parts as follows.

$$x\to 1^{+} \Rightarrow y\to +\infty$$
$$x\to 1^{-} \Rightarrow y\to -\infty$$

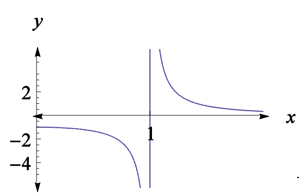


Figure 2.24

$$x\to -1^{+} \Rightarrow y\to -\infty$$
$$x\to -1^{+} \Rightarrow y\to -\infty$$

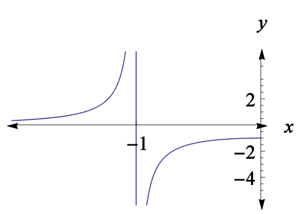
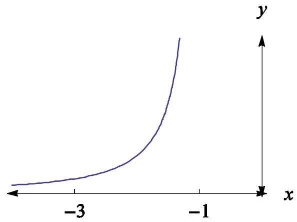


Figure 2.25

$x \to \pm\infty \Rightarrow y \to 0^+$



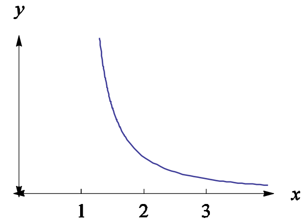


Figure 2.26 Figure 2.27

Noting the $y-$intercept is $(0,-1)$, the full graph is,

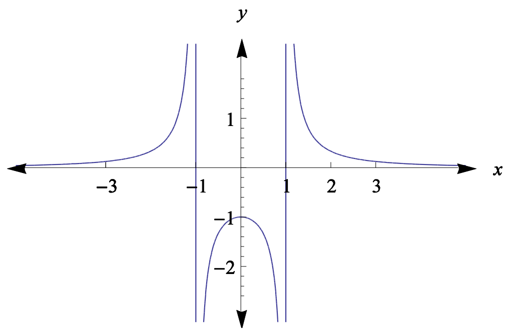


Figure 2.28 $y = \dfrac{1}{(x-1)(x+1)}$, $x \neq \pm 1.$

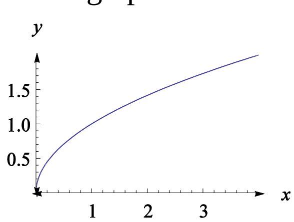
Example 4

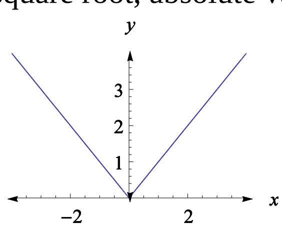
Put $y = \dfrac{2x+3}{x-1}$ in standard form and state the asymptotes.

<u>Solution;</u> By long division, $y = \dfrac{2x+3}{x-1} = 2 + \dfrac{1}{x-1}$. The asymptotes are $x = 1, y = 2$.

2.3.3 Other Basic Graphs

Other graphs include the square root, absolute value and greatest integer functions.





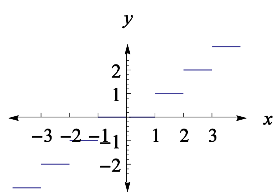


Figure 2.29 $y = \sqrt{x}$ Figure 2.30 $y = |x|$ Figure 2.31 $y = [x]$

Notes: (1) The graph of $y = |x|$ passes through $(1,1)$ as well as through $(-1,1)$. For $x \geq 0$ it is the graph of $y = x$. For $x \leq 0$, it is the graph of $y = -x$.

(2) The graph of $y = \sqrt{x}$ also passes through $(1,1)$. You can get a better idea of its graph by also plotting $(4,2)$.

2.3.4 Conic sections

The other common graphs you have encountered are the graphs of the conic sections - circles, parabolas, ellipses and hyperbolas.

Circle: A circle is defined to be the set of all points equidistant from a given fixed point or center. In the case of the distance being r and the center being the origin $(0,0)$, the equation is $x^2+y^2=r^2$ and a sample graph of $x^2+y^2=9$ is shown in Figure 2.32.

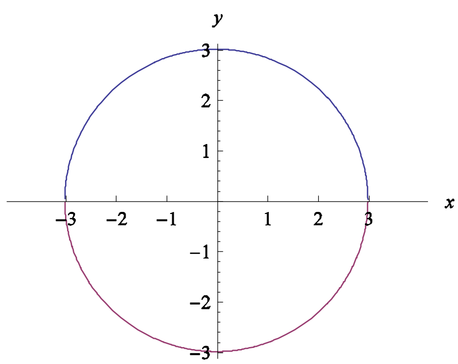


Figure 2.32: $x^2+y^2=9$

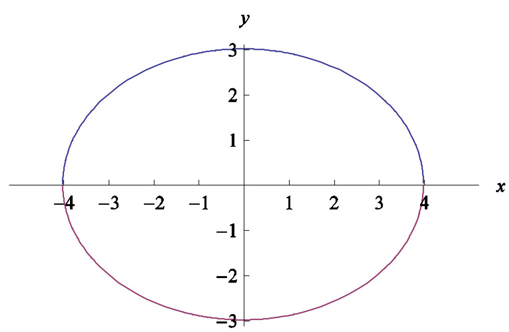


Figure 2.33 $\frac{x^2}{16}+\frac{y^2}{9}=1$

Ellipse: An ellipse is defined to be the set of all points for which the sum of their distances from two fixed points (the focal points) is a fixed distance $2a$. The equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The vertices and ends of the major axis are $(\pm a,0)$ if $a>b$ or $(0,\pm b)$ if $a<b$. The ends of the minor axis are $(0,\pm b)$ if $a>b$ or $(\pm a,0)$ if $a<b$.

For $a>b$, the focal points are $(\pm c,0)$ where c is given by given by $c^2=a^2-b^2$. For $a<b$, the focal points are $(0,\pm c)$, where c is given by given by $c^2=b^2-a^2$. A sample graph for $\frac{x^2}{16}+\frac{y^2}{9}=1$ is shown in Figure 2.33.

Example 5

Sketch the graph of $5x^2+36y^2=180$ and find the vertices, ends of the major and minor axes and the foci.

<u>Solution:</u> Divide by 180 to get $\frac{x^2}{36}+\frac{y^2}{5}=1 \Rightarrow a=6, b=\sqrt{5}, c=\sqrt{a^2-b^2}=\sqrt{31}$.

Ends of major axes are $(\pm 6,0)$, ends of the minor axes are $(0,\pm\sqrt{5})$, foci are $(\pm\sqrt{31},0)$.

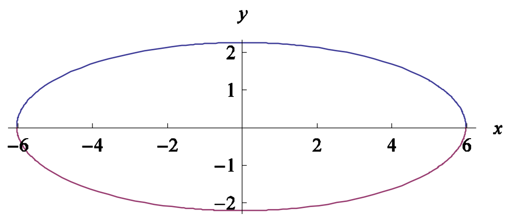


Figure 2.33b: $5x^2 + 36y^2 = 180$

Hyperbola: A hyperbola is defined to be the set of all points for which the difference of their distances from two fixed points (the focal points $(\pm c, 0)$) is a fixed distance $2a$. The equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where we define b by $b^2 = c^2 - a^2$. To draw the graph we first draw the box with vertices $(\pm a, \pm b)$. The asymptotes are $y = \pm \frac{b}{a} x$. They form the major diagonals of the box. Then we draw the hyperbola so its vertices are $(\pm a, 0)$ on the $x-axis$. The graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$, asymptotes $y = \pm 2x/3$, vertices $(\pm 3, 0)$ is shown in Figure 2.34. Note, the graph of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is symmetrical about the $y-$axis with foci $(0, \pm c)$ and vertices $(0, \pm b)$. The asymptotes remain $y = \pm \frac{b}{a} x$.

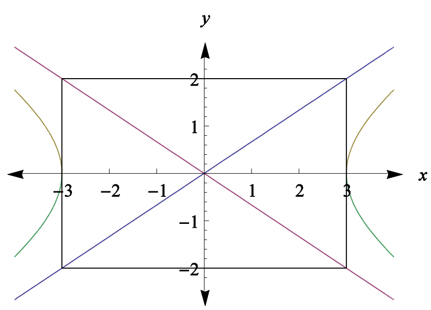


Figure 2.34 $\frac{x^2}{9} - \frac{y^2}{4} = 1$

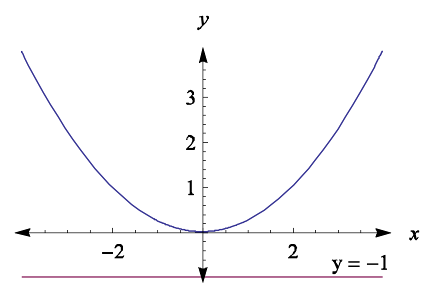


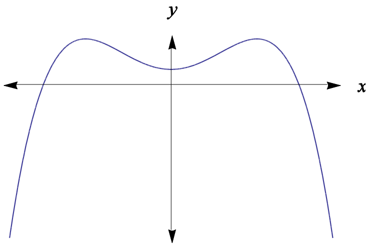
Figure 2.35 $x^2 = 4y,\ p = 1$

Parabola: A parabola is defined somewhat differently. It is the set of all points which are the same distance from a fixed point, the focus, say $(0, p)$ as they are from a fixed line, the directrix, say $y = -p$. The equation is $x^2 = 4py$ and the graph of $x^2 = 4y,\ p = 1$ is shown in Figure 2.35 above.

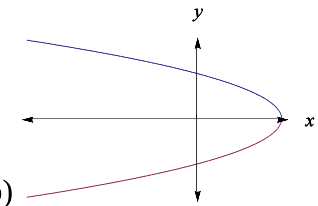
There are similar equivalent definitions for the ellipse and hyperbola in terms of the ratio (eccentricity e) of the distance from a fixed focal point and a fixed directrix. For a parabola, $e = 1$, for an ellipse, $e < 1$, and for a hyperbola, $e > 1$. For both the ellipse and the hyperbola with the foci on the $x-axis$, it turns out that $e = c/a$.

Exercises 2A

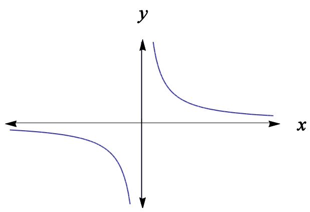
1. Use the vertical line test to determine whether the curve is the graph of a function.



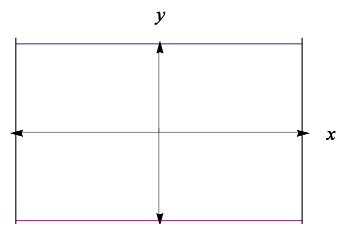
(a)



(b)

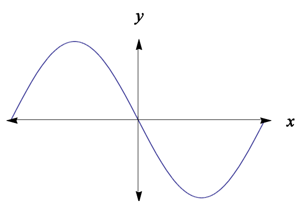


(c)

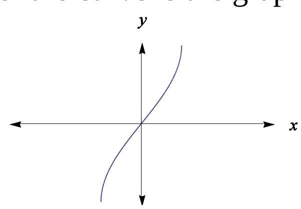


(d)

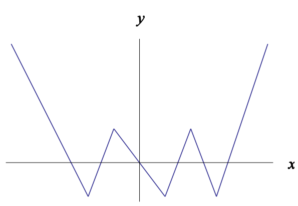
2. Use the vertical line test to determine whether the curve is the graph of a function.



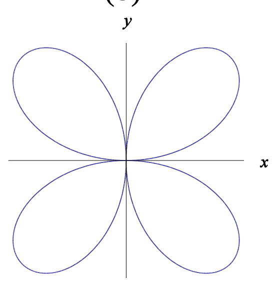
(a)



(b)



(c)



(d)

3. Does $y=(x-1)^2$ define y as a function of x?
4. Does $x^2+(y-1)^2=4$ define y as a function of x?
5. Complete the tables for $f(x)=\dfrac{x}{x-2}$. Determine the asymptotes.

x	$f(x)$
1.5	
1.9	
1.99	
1.999	

x	$f(x)$
2.5	
2.1	
2.01	
2.001	

x	$f(x)$
10	
50	
100	
1000	

x	$f(x)$
-10	
-50	
-100	
-1000	

6. Find the horizontal and vertical asymptotes (if any) of $f(x) = \dfrac{1}{x-2}$

Sketch the graph of each of the equations in Problems 7 through 13.

7. $f(x) = \dfrac{1}{x-1}$
8. $f(x) = \dfrac{x+2}{x+3}$
9. $f(x) = \dfrac{3}{(x+2)(x-4)}$
10. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$
11. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $9x^2 + 3y^2 = 27$
12. Find the vertices, foci, equations of the asymptotes of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$
13. Find the vertices, foci, equations of the asymptotes of the hyperbola $4x^2 - y^2 = 4$

==

Let us now review the operation of composition of functions from your algebra courses.

2.4 Composition of functions

2.4.1 Definition

The composition of the functions f and g is $f \circ g(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Example 6
Given $f(x) = x+2$ and $g(x) = 4 - x^2$, find $(a)\,(f \circ g)(x)$ $(b)\,(g \circ f)(x)$ $(c)\,(g \circ f)(-2)$
<u>Solution</u>
(a) $(f \circ g)(x) = f(g(x)) = g(x) + 2 = 4 - x^2 + 2 = -x^2 + 6$
(b) $g(f(x)) = 4 - [f(x)]^2 = 4 - [x+2]^2 = -x^2 - 4x$
(c) $g(f(-2)) = -(-2)^2 - 4(-2) = 4$

Example 7
Given $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{2}{x-1}$, find (a) $(f \circ g)(x)$ and its domain, (b) $g(f(x))$ and its domain.

Solution

(a) $(f \circ g)(x) = f(g(x)) = \dfrac{g(x)}{g(x)+1} = \dfrac{\dfrac{2}{x-1}}{\dfrac{2}{x-1}+1} = \dfrac{2}{x+1}$

x must first be in the domain of g which is $\{x : x \neq +1\}$ and then $g(x)$ must be in the domain of f which is $\{x : x \neq -1\}$. Since $g(x) \neq -1 \Leftrightarrow \dfrac{2}{x-1} \neq -1 \Leftrightarrow x \neq -1$, the domain of $f \circ g(x)$ is therefore $\{x : x \neq \pm 1\}$.

(b) $(g \circ f)(x) = g(f(x)) = \dfrac{2}{f(x)-1} = \dfrac{2}{\dfrac{x}{x+1}-1} = -2x-2$

x must first be in the domain of f which is $\{x : x \neq -1\}$ and then $f(x)$ must be in the domain of g which is $\{x : x \neq 1\}$. Since $f(x) \neq 1 \Leftrightarrow \dfrac{x}{x+1} \neq 1$ is always true, the domain of $g \circ f(x)$ is therefore $\{x : x \neq -1\}$.

Example 8

Given $f(x) = \sqrt{x}$, $g(x) = 2x-1$, find (a) $(f \circ g)(x)$ and its domain, (b) $g(f(x))$ and its domain.

Solution

(a) $f(g(x)) = \sqrt{g(x)} = \sqrt{2x-1}$

x must be in the domain of g which is $\mathbb{R}$ and then $g(x)$ must be in the domain of f which is $\{x : x \geq 0\}$. Since $g(x) \geq 0 \Leftrightarrow 2x-1 \geq 0 \Leftrightarrow x \geq \dfrac{1}{2}$, the domain of $f \circ g(x)$ is therefore $\mathbb{R} \cap \{x : x \geq \dfrac{1}{2}\} = \{x : x \geq \dfrac{1}{2}\}$.

(b) $g(f(x)) = 2f(x) - 1 = 2\sqrt{x} - 1$

x must be in the domain of f which is $\{x : x \geq 0\}$ and then $f(x)$ must be in the domain of g which is $\mathbb{R}$. The domain is therefore $\mathbb{R} \cap \{x : x \geq 0\} = \{x : x \geq 0\}$.

2.5 Inverse Functions

We now add to our repertoire of functions by defining, where possible, inverse functions.

2.5.1 Definition

Two functions f and g are said to be inverse functions if and only if the compositions $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f

Example 9

Prove $f(x)=\frac{1}{3}x-7$ and $g(x)=3x+21$ are inverse functions.

<u>Solution:</u> $f(g(x))=\frac{1}{3}g(x)-7=\frac{1}{3}(3x+21)-7=x$

and $g(f(x))=3f(x)+21=3(\frac{1}{3}x-7)+21=x$

So f and g are inverse functions.

2.5.2 Notation

We write a function and its inverse as f and f^{-1} or g and g^{-1}, etc. Then our definition becomes: Two functions f and f^{-1} are said to be inverse functions if and only if $f(f^{-1}(x))=x$ for all x in the domain of f^{-1} and $f^{-1}(f(x))=x$ for all x in the domain of f.

2.5.3 Finding inverse functions

Assuming the inverse function of $y=f(x)$ exists, we can find it as follows:

1. Interchange x and y in the equation $y=f(x)$ to get $x=f(y)$.
2. Solve the new equation for y in terms of x. This new equation is the inverse function $y=f^{-1}(x) \Leftrightarrow x=f(y)$.

Note, since we have interchanged x and y it follows that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

Example 10

Find the inverse function of $f(x)=4x-3$

<u>Solution:</u>

Interchange the variables in $y=4x-3$ to get $x=4y-3$

Solve for y: $y=f^{-1}(x)=\frac{x+3}{4}$

(Check $f(f^{-1}(x))=4f^{-1}(x)-3=4\frac{(x+3)}{4}-3=x$. You can check the other condition.

Example 11

Find the inverse function of $y=f(x)=\frac{x-1}{x+1}$

<u>Solution</u>

Interchange to get

$$x=\frac{y-1}{y+1} \Leftrightarrow xy+x=y-1 \Leftrightarrow y-xy=x+1 \Leftrightarrow y(1-x)=1+x \Leftrightarrow y=f^{-1}(x)=\frac{1+x}{1-x}$$

2.5.4 Sketching the graphs of inverse functions

Since we interchanged variables to obtain the inverse function, it follows that if (a,b) lies on the graph of a function, then (b,a) lies on the graph of its inverse. This means the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$ as shown in Figure 2.36. Note, the point (b,a) on the graph of $y = f(x)$ reflects to become the point (a,b) on the graph of $y = f^{-1}(x)$ as follows. From (b,a) on the graph of $y = f(x)$, you measure the shortest distance to a point on the line $y = x$ and then go the same distance in the same direction on the other side of $y = x$ to locate (a,b) on the graph of $y = f^{-1}(x)$.

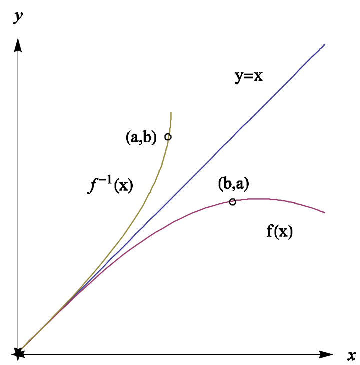


Figure 2.36

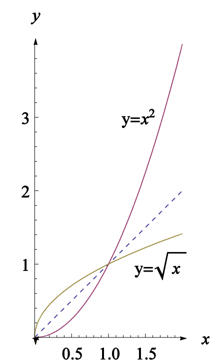


Figure 2.37 $f(x) = x^2, f^{-1}(x) = \sqrt{x}, x > 0$

Example 12

Sketch the graph of the inverse function of $f(x) = x^2$ for $x > 0$.

Solution: See Figure 2.37.

2.5.5 When do inverse functions exist?

If we try to sketch the graph of the inverse function of $y = x^2$ shown in Figure 2.38, the problem is immediately obvious. The graph of the inverse "function" $x = y^2$ shown in Figure 2.39 fails the vertical line test, so it is NOT a function.

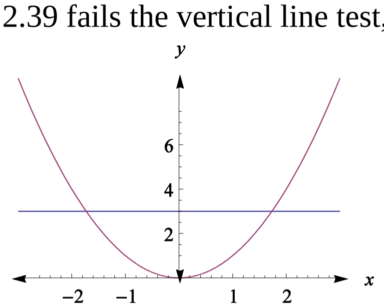


Figure 2.38 $y = x^2$

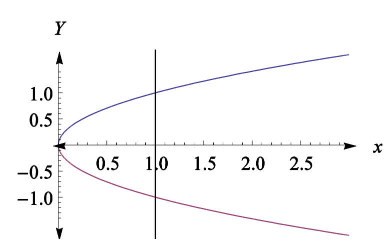


Figure 2.39 $x = y^2$

A little thought tells us that a function will therefore have an inverse function only if it satisfies the:

Horizontal line test – The inverse function f^{-1} exists if and only if any line parallel to the $x-axis$ intersects the graph of the function f at most once.

The graph of $y = x^2$ in Figure 2.38 or Figure 2.40 fails the horizontal line test and does not have an inverse for all its domain, but the graph of $y = x^3$ in Figure 2.41 passes the horizontal line test and has an inverse for all of its domain.

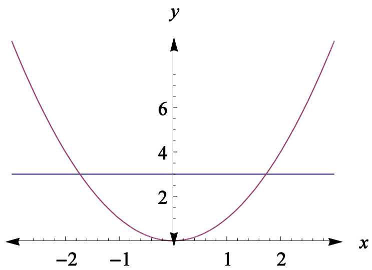


Figure 2.40 $y = x^2$

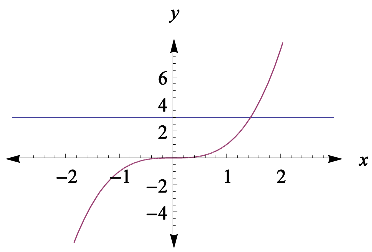


Figure 2.41 $y = x^3$

Technically, we say the function must be a one-to-one function, defined as follows:

Definition
A function is **one-to-one** if each value of the dependent variable corresponds to exactly one value of the independent variable.

We can write this algebraically as either a statement or its equivalent contrapositive[2] statement.

Statement: A function $f(x)$ is said to be one-to-one if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
Contrapositive Statement: A function $f(x)$ is said to be one-to-one if and only if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

You should satisfy yourself that the statement and the contrapositive are equivalent. Both mean that different $x-values$ necessarily give different $y-values$.

Graphically, a function $f(x)$ is said to be one-to-one if and only if it satisfies the horizontal line test.

---

[2] In Logic, if P nd Q are statements and $P \Rightarrow Q$ is true, then the contrapositive statement $not\ Q \Rightarrow not\ P$ is also true. Note, $not\ x_1 \neq x_2$ is $x_1 = x_2$.

Since it must further satisfy the vertical line test to be a function, a relation is a function with an inverse if and only if it satisfies both the vertical and horizontal line tests.

2.5.6 Functions that fail the horizontal line test

If a function fails the horizontal line test, then it does not have an inverse function *for that domain.* If we can restrict the domain so that it does satisfy the horizontal line test (so that it is one-to-one), then we can find the inverse function which will have a range the same as this restricted domain. Again, note it is important that a function be defined both by the relationship between the variables and by a nominated domain.

Example 13

Restrict the domain of $y = x^2$ so that the new function has an inverse.

Solution: Consider $y = x^2$ with the restricted domain $[0, \infty)$ or $x \geq 0$. It is now one-to-one, so it has an inverse. The inverse function must be chosen from $x = y^2 \Leftrightarrow y = \pm\sqrt{x}$. It is obviously $y = \sqrt{x}$ which now has the range $[0, \infty)$ or $y \geq 0$. See Figures 2.42, 2.43.

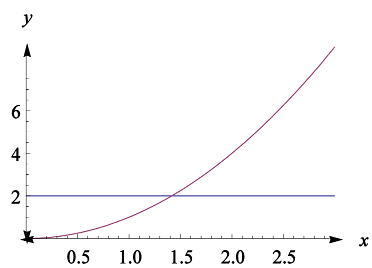


Figure 2.42 $y = x^2, x \geq 0$

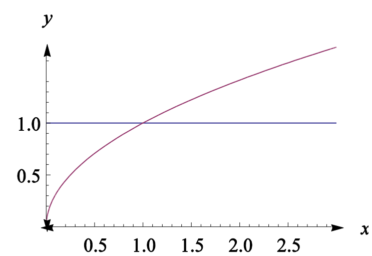


Figure 2.43 $y = \sqrt{x}, x \geq 0$

Exercises 2B

1. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = 2x, \quad g(x) = 3x - 1$
2. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = 3x - 4, \quad g(x) = x^2 + 3x - 4$
3. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 7$
4. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = \sqrt{x - 2}, \quad g(x) = 3x - 1$
5. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = \dfrac{1}{x - 1}, \quad g(x) = \dfrac{2}{x}$
6. Find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and specify the domain of each for $f(x) = 2x + 1, \quad g(x) = \sqrt{x - 1}$

7. If $f(x) = x^2 - 2, \ g(x) = x + 4$, find $(f \circ g)(2)$ *and* $(g \circ f)(-4)$.
8. If $f(x) = x - 4$, find $f^{-1}(x)$ and verify $(f \circ f^{-1})(x) = x$ *and* $(f^{-1} \circ f)(x) = x$
9. If $f(x) = \dfrac{3}{4}x - \dfrac{5}{6}$, find $f^{-1}(x)$ and verify $(f \circ f^{-1})(x) = x$ *and* $(f^{-1} \circ f)(x) = x$
10. If $f(x) = \sqrt{x}, \ x \ge 0$, find $f^{-1}(x)$ and verify $(f \circ f^{-1})(x) = x$ *and* $(f^{-1} \circ f)(x) = x$
11. If $f(x) = \dfrac{2}{x - 1}, x \ge 1$, find $f^{-1}(x)$ and graph f *and* f^{-1} on the same set of axes.
12. Restrict the domain of $f(x) = |x - 5|$ so that the function is one-to-one and has an inverse. Then find the inverse function f^{-1} and state the domain and range of f *and* f^{-1}.
13. Restrict the domain of $f(x) = (x - 4)^2$ so that the function is one-to-one and has an inverse. Then find the inverse function f^{-1} and state the domain and range of f *and* f^{-1}.
14. Restrict the domain of $f(x) = \dfrac{1}{2}x^2 - 1$ so that the function is one-to-one and has an inverse. Then find the inverse function f^{-1} and state the domain and range of f *and* f^{-1}.

Chapter 3

Transformations of Functions and Relations

3.1 Translation Rule

The translation rule applies to translating or shifting the graph of a function so that the orientation and shape of the graph do not change.

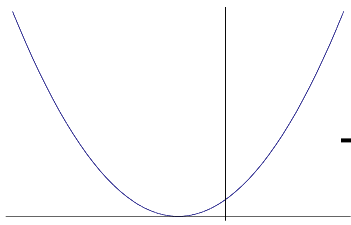


Figure 3.1

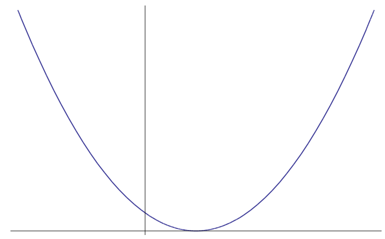


Figure 3.2

<u>Translation Rule</u>

The graph of $y = f(x)$ ***maintains the same shape and orientation under the translation*** $y-k = f(x-h)$. ***The origin*** $(0,0)$ ***shifts to the point given by*** $x-h=0,\ y-k=0$***, that is,*** $(0,0)\to(h,k)$. ***Every other point*** (x,y) ***on the graph similarly shifts to*** $(x+h, y+k)$.
Note that h and k may be positive or negative numbers or zero. If h is positive, the basic graph shifts horizontally h units to the right; if h is negative, the basic graph shifts horizontally h units to the left. If k is positive the basic graph shifts k units vertically up; if k is negative, the basic graph shifts k units vertically down.

Example 1

Sketch $y=(x-1)^2+2$

<u>Solution:</u> Rewrite the equation as $y-2=(x-1)^2$

The basic shape is that of $y=x^2$ with the origin shifted to $x-1=0,\ y-2=0$ or $(0,0)\to(1,2)$ and all other points on the graph of $y=x^2$ shift accordingly. That is, the basic graph shifts horizontally 1 unit to the right and vertically 2 units up.

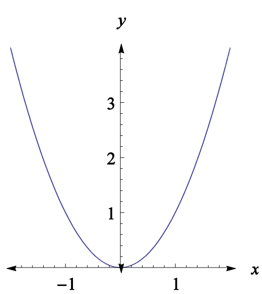


Figure 3.3 $y=x^2$

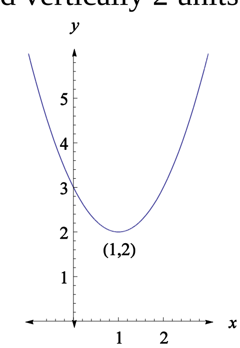
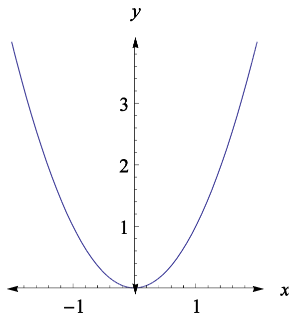


Figure 3.4 $y=(x-1)^2+2$

Example 2

Sketch $y+3=(x-1)^2$

<u>Solution:</u> The basic shape is that of $y=x^2$ translated so the origin goes to $x-1=0,\ y+3=0$, that is $(0,0)\to(1,-3)$. The basic graph shifts horizontally 1 unit to the right and vertically 3 units down.



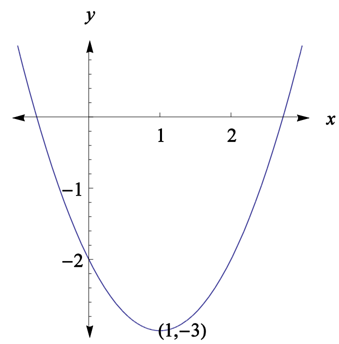


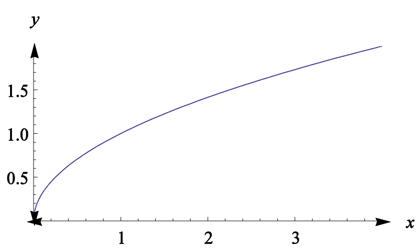
Figure 3.5 $y=x^2$

Figure 3.6 $y+3=(x-1)^2$

Example 3

Sketch $y=\sqrt{x-2}\Leftrightarrow y-0=\sqrt{x-2}$

<u>Solution:</u> The basic shape is that of $y=\sqrt{x}$ translated so $(0,0)\to(2,0)$. The basic graph shifts 2 units to the right.



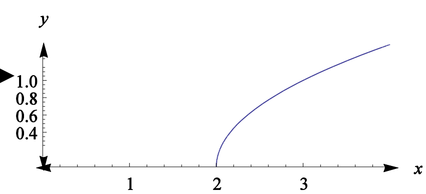


Figure 3.7 $y=\sqrt{x}$

Figure 3.8 $y=\sqrt{x-2}$

Now we recall the treatment of asymptotes in Section 2.3.2. The alternative method of drawing graphs of rational functions where the degree of the denominator is 1 is to use the transformation rules.

Example 4

Sketch $y=\dfrac{1}{x-1}-3\Leftrightarrow y+3=\dfrac{1}{x-1}$

<u>Solution:</u> The basic shape is that of $y=\dfrac{1}{x}$ translated so that $(0,0)\to(1,-3)$. In particular, since the asymptotes were $x=0,\ y=0$ or the two axes, the new asymptotes will be $x=1,\ y=-3$. The basic graph shifts horizontally 1 unit to the right and vertically 3 units down.

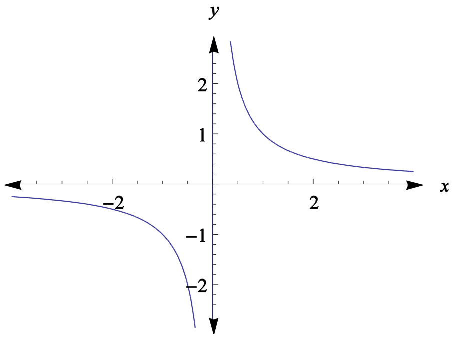


Figure 3.9 $y = \dfrac{1}{x}$

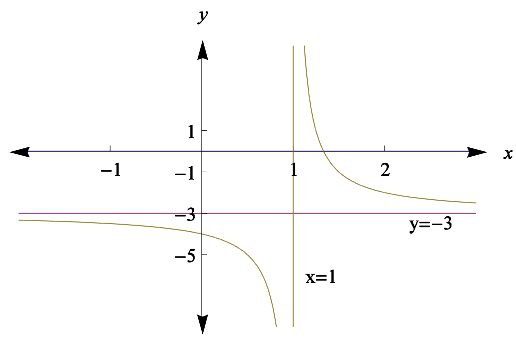


Figure 3.10 $y + 3 = \dfrac{1}{x-1}$

Now we note that the translation rule applies to the graphs of equations of relations, not only functions, as we see in the following.

Example 5

Sketch $(x-1)^2 + (y-2)^2 = 4$

<u>Solution:</u> The basic shape is that of $x^2 + y^2 = 2^2$, a circle center $(0,0)$ and radius 2. This is to be shifted horizontally 1 unit right and vertically 2 units up. The translation is $(0,0) \to (1,2)$ which becomes the new center. The radius is unaltered.

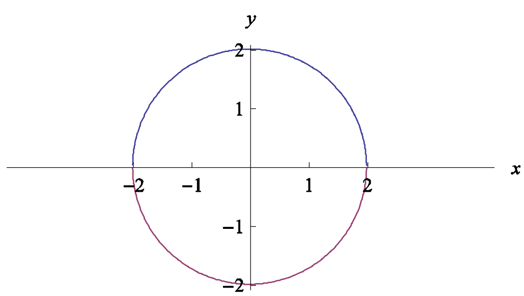


Figure 3.11 $x^2 + y^2 = 2^2$

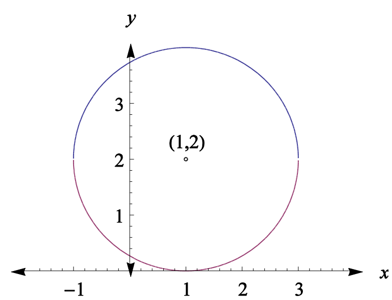


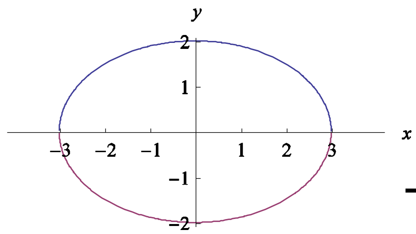
Figure 3.12 $(x-1)^2 + (y-2)^2 = 4$

Example 6

Sketch $\dfrac{(x+3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

<u>Solution:</u> The basic shape is that of the ellipse $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$ which is an ellipse centered at the origin $(0,0)$ with $(\pm 3, 0)$ and $(0, \pm 2)$ as the end points of its major and minor axes respectively. The translation of these five points is as follows. Note the lengths of the major and minor axes remain the same.

$$
\begin{aligned}
&(0,0) &&\to (-3,-2) \\
&(3,0) &&\to (3-3,0-2) = (0,-2) \\
&(-3,0) &&\to (-3-3,0-2) = (-6,-2) \\
&(0,2) &&\to (0-3,2-2) = (-3,0) \\
&(0,-2) &&\to (0-3,-2-2) = (-3,-4)
\end{aligned}
$$



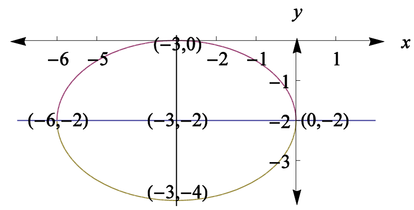


Figure 3.13 $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ Figure 3.14 $\frac{(x+3)^2}{9}+\frac{(y+2)^2}{4}=1$

Example 7

Sketch the graph of $4x^2+9y^2+24x+36y+36=0$

Solution: We complete the square on both the x and y terms.

$$
\begin{aligned}
&4x^2+9y^2+24x+36y+36=0 \\
&4(x^2+6x)+9(y^2+4y)+36=0 \\
&4(x^2+6x+9-9)+9(y^2+4y+4-4)+36=0 \\
&4(x+3)^2+9(y+2)^2=36 \\
&\frac{(x+3)^2}{9}+\frac{(y+2)^2}{4}=1
\end{aligned}
$$

Then sketch the graph of the ellipse as in the previous example.

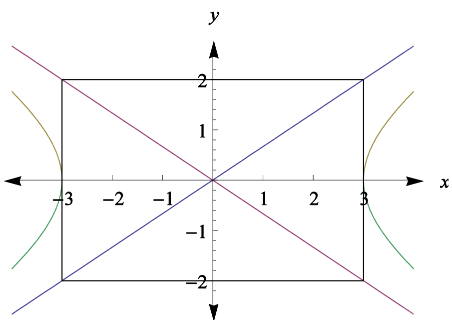
Example 8

Sketch $\frac{(x-2)^2}{9}-\frac{(y-3)^2}{4}=1$

Solution: The basic graph is that of the hyperbola $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$ which is drawn by first drawing the box with vertices $(\pm 3,\pm 2)$, the asymptotes $y=\pm\frac{2}{3}x$ which are the diagonals of the box, and then the hyperbola with vertices $(\pm 3,0)$. The translation takes:

$$
\begin{aligned}
&(0,0) &&\to (2,3) \\
&(-3,2) &&\to (-3+2,2+3) = (-1,5) \\
&(3,2) &&\to (3+2,2+3) = (5,5) \\
&(-3,-2) &&\to (-1,1)
\end{aligned}
$$

$$
\begin{aligned}
&(3,-2) &&\to (5,1) \\
&(-3,0) &&\to (-1,3) \\
&(3,0) &&\to (5,3) \\
&y=\pm\frac{2}{3}x &&\to (y-3)=\pm\frac{2}{3}(x-2)
\end{aligned}
$$



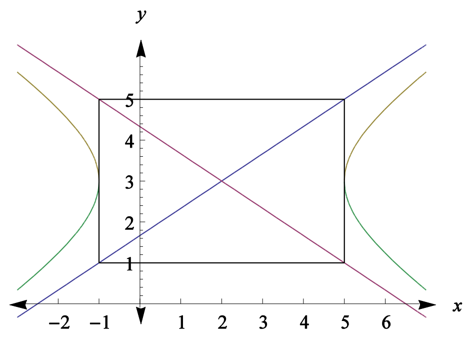


Figure 3.15 $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$ Figure 3.16 $\frac{(x-2)^2}{9}-\frac{(y-3)^2}{4}=1$

The same rules apply to all of our basic graphs.

Example 9

Sketch the graph of $y=|x-2|+1$

<u>Solution</u>

The graph of $y-1=|x-2|$ has the same shape and orientation as that of $y=|x|$ except that the latter is translated so $(0,0)\to(2,1)$ which becomes the new vertex, or the basic graph shifts 2 units to the right and up 1 unit. The $y-$intercept is $y=1+|0-2|=3$. We can use the symmetry of the graph to determine a third point $(4,3)$.

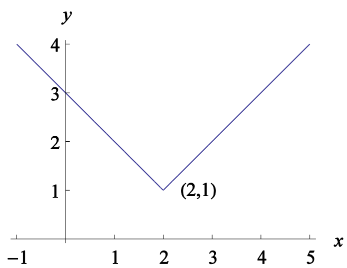


Figure 3.17 $y=|x-2|+1$

Exercises 3A

Translation Rule

1. Graph $f(x)=x^2+1$
2. Graph $f(x)=(x+2)^2$
3. Sketch $f(x)=\sqrt{x+4}$
4. Sketch $y=x^2+3$

5. Sketch $f(x)=\sqrt{x}+1$
6. Sketch $f(x)=|x|-1$
7. Sketch $f(x)=\sqrt{x+4}-3$
8. Find the center and radius of the circle $x^2+y^2=9$ and sketch its graph.
9. Find the center and radius of the circle $(x+3)^2+(y-4)^2=25$ and sketch its graph.
10. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$
11. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{4}=1$
12. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $x^2+6x+9y^2-36y+36=0$
13. Find the vertices, endpoints of the major and minor axes and the foci of the ellipse $16x^2+9y^2+36y-108=0$
14. Find the vertices, foci, equations of the asymptotes and sketch the hyperbola $4y^2-x^2=4$
15. Find the vertices, foci, equations of the asymptotes and sketch the hyperbola $x^2-y^2=9$
16. Find the vertices, foci, equations of the asymptotes and sketch the hyperbola $\frac{(x-1)^2}{9}-\frac{(y+4)^2}{4}=1$
17. Find the vertices, foci, equations of the asymptotes and sketch the hyperbola $9x^2+72x-4y^2-16y+92=0$

==================================

3.2 Reflection Rules

The Reflection rules are:

(a) The graph of $y=-f(x)$ ***is obtained by reflecting the graph of*** $y=f(x)$ ***in the*** $x-axis$.

(b) The graph of $y=f(-x)$ ***is obtained by reflecting the graph of*** $y=f(x)$ ***in the*** $y-axis$.

Example 10: Example of Rule (a)

Sketch $y=x^2$ and $y=-x^2$

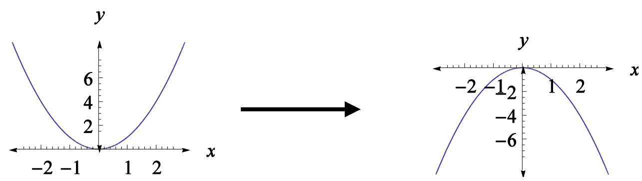
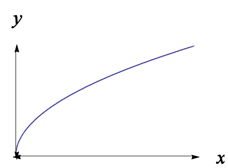


Figure 3.18 $y = x^2 \to y = -x^2$

Example 11: Example of Rule (b)

Graph $y = \sqrt{x}$ and $y = \sqrt{-x}$

Note the domain of the first is $x \geq 0$ and of the second is $x \leq 0$.



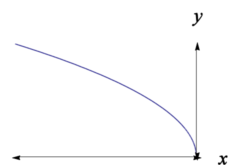


Figure 3.19 $y = \sqrt{x}$

Figure 3.20 $y = \sqrt{-x}$

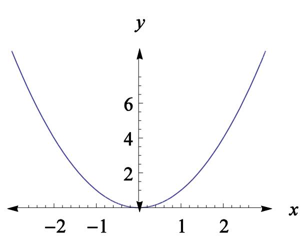
3.3 Combining the translation and reflection rules

Do the reflection first and the translation second.

Example 12

Graph $y - 3 = -(x-2)^2$

<u>Solution:</u> The basic shape is that of $y = x^2$ reflected in the $x-axis$ and translated so $(0,0) \to (2,3)$.



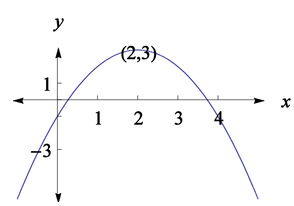


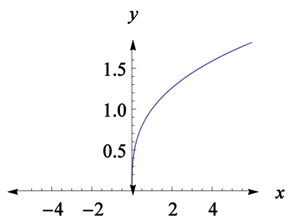
Figure 3.21 $y = x^2$

Figure 3.22 $y - 3 = -(x-2)^2$

Example 13

Sketch $y = \sqrt[3]{2-x} = \sqrt[3]{-(x-2)}$

<u>Solution:</u> The basic shape is that of $y = \sqrt[3]{x}$ reflected in the $y-axis$ and translated so $(0,0) \to (2,0)$.



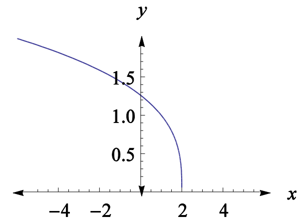


Figure 3.23 $y = \sqrt[3]{x}$ Figure 3.24 $y = \sqrt[3]{2-x}$

3.4 Vertical Stretching and Shrinking Rules

The vertical stretching and shrinking rules are:
Let f be a function and a be a positive real number.

(a) If $|a| > 1$, the graph of $y = a\,f(x)$ is the graph of $y = f(x)$ vertically stretched by multiplying each of its $y-$coordinates by a.

(b) If $|a| < 1$, the graph of $y = a\,f(x)$ is the graph of $y = f(x)$ vertically shrunk by multiplying each of its $y-$coordinates by a.

Example 14: Example of Rule (a)

Sketch $y = 2x^2$

<u>Solution</u> Figure 3.25.

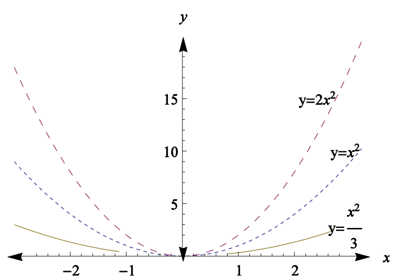


Figure 3.25 Graphs of $y = 2x^2, x^2, \frac{1}{3}x^2$

Example 15: Example of Rule (b)

Sketch $y = \frac{1}{3}x^2$

<u>Solution</u> Figure 3.25 above.

3.5 Horizontal Stretching and Shrinking Rules

The *horizontal stretching and shrinking rules are:*
Let f be a function and b be a positive real number.

(c) If $|b| > 1$, the graph of $y = f(bx)$ is the graph of $y = f(x)$ horizontally shrunk by dividing each of its $x-$coordinates by b.

(d) If $|b| < 1$, the graph of $y = f(bx)$ is the graph of $y = f(x)$ horizontally stretched by dividing each of its $x-$coordinates by b.

Example 16: Example of Rule (a)

If $y = f(x) = -x^2$, sketch $y = f(2x) = -(2x)^2$.

Solution See Figure 3.26. We shrink the graph of $y = -x^2$.

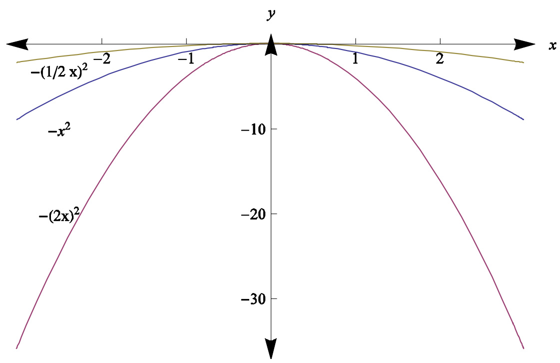


Figure 3.26

Example 17: Example of Rule (b)

If $y = f(x) = -x^2$, sketch $y = f\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2$.

Solution

See Figure 3.26 above. We stretch the basic graph.

Exercises 3B

Reflection rule

1. Graph $f(x) = -x^2 - 2x + 1$
2. Graph $f(x) = -x^3$
3. Graph $f(x) = -2\sqrt{x}$

Reflection and translation rules

4. Graph $f(x) = \sqrt{2 - x}$
5. Graph the quadratic function $f(x) = -x^2 + 2$
6. Graph the quadratic function $f(x) = (x-1)^2 + 2$
7. Graph $f(x) = |x+1| - 3$

8. Graph $f(x)=\dfrac{4x}{x-1}$
9. Graph $f(x)=\dfrac{-1}{x-3}$
10. Sketch $y=(x-4)^2+1$
11. Sketch $y=-\sqrt{x+2}$
12. Sketch $y=|x+2|$
13. Sketch $y=\dfrac{1}{x-3}-1$
14. Sketch $y=-x^3+4$

Vertical and horizontal stretching and shrinking

15. Sketch $f(x)=2(x-7)^2$
16. Sketch $f(x)=-\dfrac{1}{4}(x+2)^2-2$
17. Sketch $f(x)=3(x-2)^3$
18. Sketch $f(x)=\dfrac{1}{2}|x-2|-3$
19. Sketch $f(x)=-2x^2$
20. Sketch $f(x)=-2(x+1)^2$
21. Sketch $f(x)=\sqrt{\dfrac{1}{4}x}$

====================================

3.6 Intercepts with the axes

Whenever we sketch a graph, we should always label any intercepts with the axes. These points generally control vertical and horizontal stretching and shrinking, thus obviating the need to consider them.

The $x-$intercepts are given by solving $y=f(x)=0.$ These are also called the **zeros** of the function. The $y-$intercepts are given by solving $y=f(0),$ that is putting $x=0.$

Example 18
Find the x and $y-$intercepts of $y=f(x)=x^2-4x-12$.
<u>Solution:</u>
$x-$intercepts: $f(x)=0\Rightarrow x^2-4x-12=0\Rightarrow(x-6)(x+2)=0\Rightarrow x=6,-2$
$y-$intercept: $x=0\Rightarrow y=f(0)=-12$
As we did previously, we present these results in the form of a table of values:

x	y
0	?
?	0

$\Rightarrow$

x	y
0	-12
$-2, 6$	0

Example 19

Graph $y = f(x) = x^2 - 4x - 12$

<u>Solution:</u> We complete the square to obtain the form $y - k = (x-h)^2$.

$$f(x) = x^2 - 4x - 12$$
$$= x^2 - 4x + 4 - 4 - 12$$
$$= (x-2)^2 - 16$$
$$\Leftrightarrow y + 16 = (x-2)^2$$

This is the graph of $y = x^2$ with $(0,0) \to (2,-16)$. The graph shows this vertex as well as the intercepts with the axes determined in the previous example.

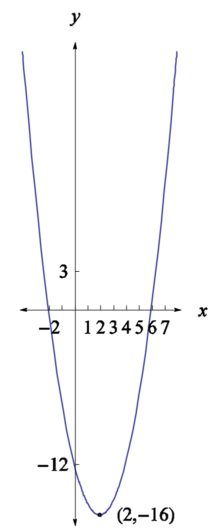


Figure 3.28 $y = f(x) = x^2 - 4x - 12$

Besides factoring we can use the completion of the square to find the zeros, for example,

$$(x-2)^2 - 16 = 0 \Rightarrow (x-2)^2 = 16$$
$$\Rightarrow x - 2 = \pm 4 \Rightarrow x = 2+4, 2-4$$
$$\Rightarrow x = -2, 6$$

We could also use the quadratic formula to find the zeros of the function, namely,

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{16+48}}{2} = -2, 6.$$

Example 20

Graph $f(x) = -3x^2 + 6x - 11$.

<u>Solution:</u> Complete the square to obtain the form $y - k = a(x-h)^2$

$$y = -3(x^2 - 2x) - 11$$
$$y = -3(x^2 - 2x + 1 - 1) - 11$$
$$y = -3(x-1)^2 + 3 - 11$$
$$y + 8 = -3(x-1)^2$$

The basic shape is that of $y = x^2$. We reflect $y = x^2$ in the $x-axis$ and translate so $(0,0) \to (1,-8)$. The "3" means we need to stretch vertically, but we can take care of this most simply by calculating the intercepts with the axes.

The table of values is as shown.

x	y
0	-11
No values	0

The calculations required for the table are:

$x = 0 \Rightarrow y = f(0) = -11$

$$y = 0 \Rightarrow x = \frac{-6 \pm \sqrt{36-132}}{-6}$$

These are complex numbers.

Note, by symmetry, the point $(2,-11)$ is also on the graph. (WebAssign may require you to plot three points including this one.)

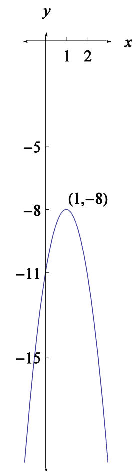


Figure 3.29 $f(x) = -3x^2 + 6x - 11$

3.7 Summary of Transformation Rules

The graph of $y - k = af(b(x-h))$ is obtained from the basic graph of $y = f(x)$ as follows.

(a) If $a < 0$, reflect in the $x-axis$.

(b) If $b < 0$, reflect in the $y-axis$.

(c) The basic graph is translated so $(0,0) \to (h,k)$, the whole graph moves accordingly.

(d) If $|a| < 1$, the basic graph is shrunk vertically by that factor, if $|a| > 1$ the basic graph is stretched vertically by that factor.

(e) If $|b| > 1$, the basic graph is shrunk horizontally by a factor of b. If $|b| < 1$, the basic graph is stretched horizontally by a factor of b.

3.8 Symmetry

It is sometimes easier to plot a graph by drawing the part of it that lies in one quadrant (often the first) and using symmetry rules to draw the other part/s. There are three types of symmetry.

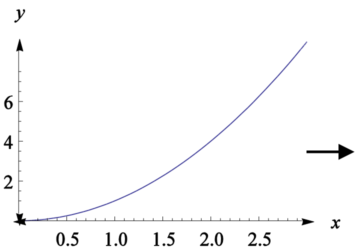
<u>$y-axis$ **symmetry**</u>

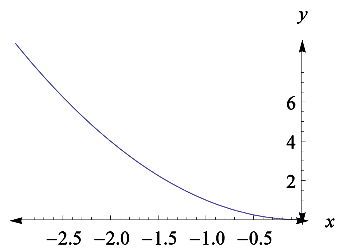
A graph exhibits $y-axis$ symmetry if it is unchanged when reflected about the $y-axis$.
This is always the case if $y = f(x)$ is the same as $y = f(-x)$, that is, when x is replaced with $-x$ in the equation of the function, the function is unchanged.

Example 21

Sketch the graph of $y = (-x)^2$

Solution: $y=(-x)^2$ is the same as $y=x^2$. The graph is symmetrical about the $y-axis$. It is only necessary to plot the part of the graph in the first quadrant – we can then use $y-axis$ symmetry to sketch the part in the second quadrant.





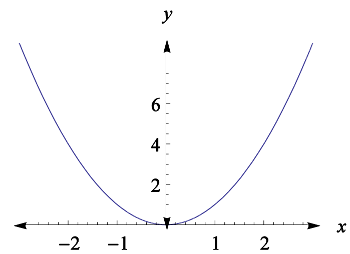


Figure 3.30 $y=(-x)^2, x\geq 0$ Figure 3.31 $y=(-x)^2, x\leq 0$ Figure 3.32 $y=(-x)^2$

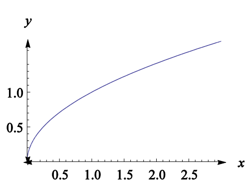
$x-axis$ symmetry

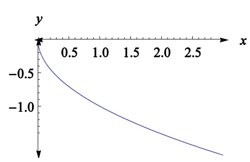
A graph exhibits $x-axis$ symmetry if it is unchanged when reflected about the $x-axis$. This is always the case if $y=f(x)$ is the same as $y=-f(x)$, that is when y is replaced with $-y$ in the equation of the function, the function is unchanged.

Example 22

Sketch the graph of $x=y^2$

Solution: $x=y^2$ is the same as $x=(-y)^2$. The graph is symmetrical about the $x-axis$. It is only necessary to plot the part of the graph in the first quadrant – we can then use $x-axis$ symmetry to sketch the part in the fourth quadrant.





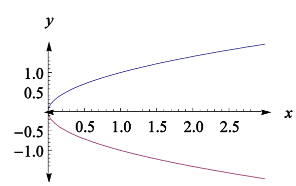


Figure 3.33 $x=y^2, y\geq 0$ Figure 3.34 $x=y^2, y\leq 0$ Figure 3.35 $x=y^2$

Origin symmetry

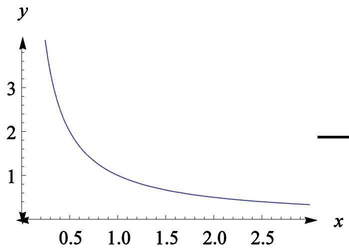
A graph exhibits origin symmetry if it is unchanged when rotated 180^0 about the origin. This is the case if the equation of the graph is unchanged when x is replaced by $-x$ and y is replaced by $-y$.

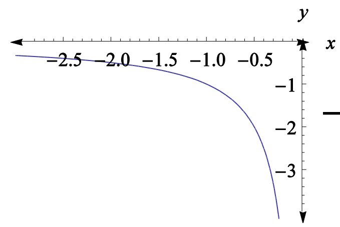
Example 23

Sketch the graph of $y=\dfrac{1}{x}$

Solution: The graph of $y=\dfrac{1}{x}$ is the same as the graph of $-y=\dfrac{1}{-x}\Leftrightarrow y=\dfrac{1}{x}$

It is only necessary to plot the part of the graph in the first quadrant – we can then use origin symmetry to sketch the part in the third quadrant.





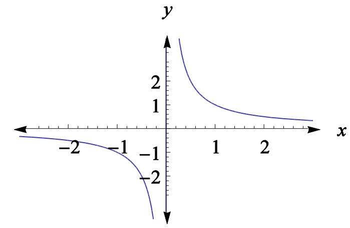


Figure 3.36 $y = \dfrac{1}{x}$, $x > 0$ Figure 3.37 $y = \dfrac{1}{x}$, $x < 0$ Figure 3.38 $y = \dfrac{1}{x}$

Example 24
Sketch the graph of $x^2 + y^2 = 4$. Show any symmetries.

<u>Solution:</u> The graph of the circle with equation $x^2 + y^2 = 4$ has all three symmetries.

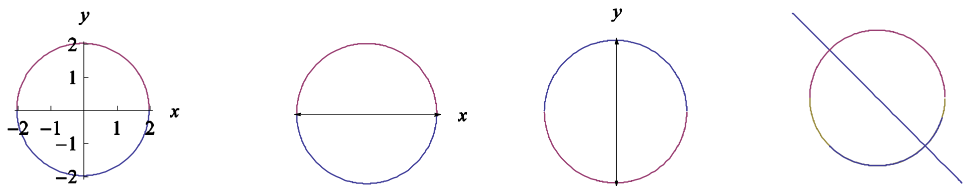


Fig. 3.39 x-axis symmetry Fig. 3.40 y-axis symmetry Fig. 3.41 Origin symmetry

3.9 Even and Odd Functions

Definition
If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an <u>even</u> function.

Example
$f(x) = x^2$ is an even function since $f(-x) = (-x)^2 = x^2 = f(x)$

The graph of an even function is symmetric with respect to the $y-axis$. This means that if we have plotted the graph for $x \ge 0$ then we can obtain the whole graph by reflecting this part in the $y-axis$. See Figures 3.30 to 3.32.

Definition
If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an <u>odd</u> function.

Example
$f(x) = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3 = -f(x)$. So is $f(x) = \dfrac{1}{x}$.

The graph of an odd function is symmetric with respect to the origin. This means that if we have plotted the graph for $x \geq 0$ then we can obtain the whole graph by rotating this part through 180^0 about the origin. See Figure 2.9 and Figures 3.36 to 3.38.

3.10 Comments on Graphs of Polynomial Functions of Degree ≥ 3

The transformation rules enable us to draw the graph of $y-k=a\,f(x-h)$ from the basic graph of $y=f(x)$. These rules apply to all polynomial functions of degree $0,1\,or\,2$, for example, $y=2,\ y-2=3(x-1),\ y=2x^2+3x-6$. They also apply to polynomials of degree ≥ 3 if they can be written in this form , e.g., $y-1=(x-2)^3,\ y+1=-(x+3)^4$, etc.

In general, however, polynomials of degree ≥ 3 cannot be written in this form. We do know, however, that their graphs are smooth, continuous curves. As $x \to \pm\infty$, the graph of $y=f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ has the same shape of that of $y=f(x)=a_nx^n$. But it generally does not have the same shape for "smaller" values of x. Consider, $f(x)=x^3$ and $f(x)=x^3+2x^2-x-2=(x-1)(x+1)(x-2)$.
See Figures 3.42 and 3.43.

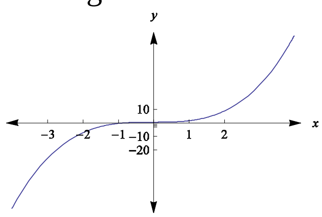


Figure 3.42 $y=f(x)=x^3$

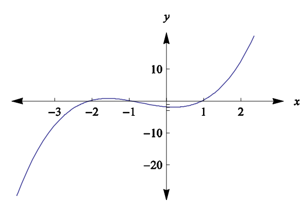
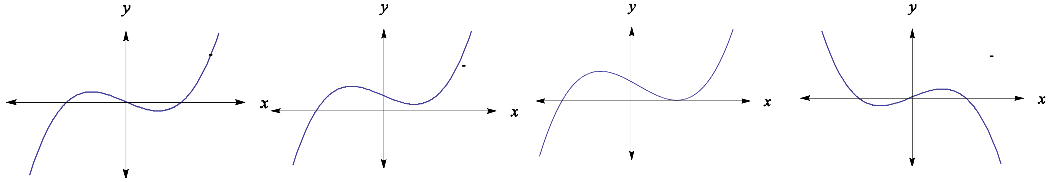


Figure 3.43 $y=f(x)=x^3+2x^2-x-2$

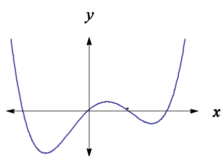
The graph of $f(x)=x^3+2x^2-x-2$ has the same shape as $x \to \pm\infty$ as $f(x)=x^3$ but it has three $x-$intercepts and two turning points (a local maximum point and a local minimum point) for "small" x. The three $x-$intercepts occur since there are always three solutions to the equation $f(x)=a_3x^3+a_2x^2+a_1x+a_0=0$ or three **zeros** of $f(x)$. For $f(x)=x^3$, they are all the same, namely 0, but for $f(x)=x^3+2x^2-x-2$ they are all different real numbers, $-2,\pm 1$. There is always one real solution, so the graph cuts the $x-axis$ at least once, but the other two solutions may be complex numbers.

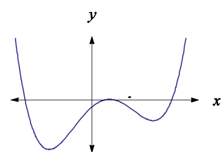
Accordingly, the graph of $f(x)=a_3x^3+a_2x^2+a_1x+a_0$ may have up to six forms,

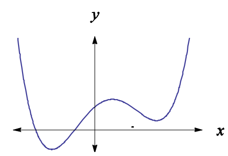


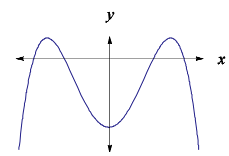
Three real zeros	One real zero	Two real zeros	Three real zeros
$a_3>0$	$a_3>0$	$a_3>0$	$a_3<0$, **etc.**

For polynomials of degree $n \geq 3$, we need to distinguish the two cases of n even and n odd since their basic graphs (see Section 2.3.1) are so distinguished. For $n = 4$, there can be as many as 4 intercepts with the $x-axis$. Some possible graphs (note the basic shape of $y = \pm x^4$ as $x \to \pm\infty$) are,

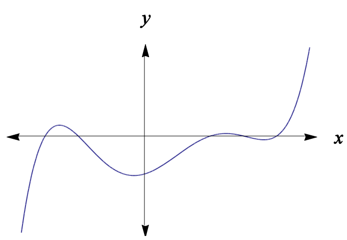




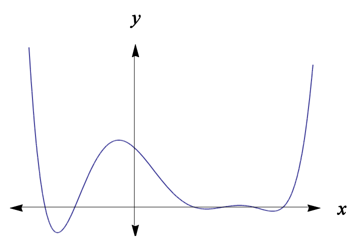




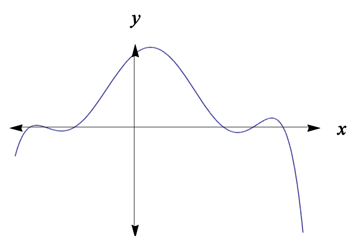
For $n = 5$ and $n = 6$, some possible graphs are,



$n = 5$



$n = 6$



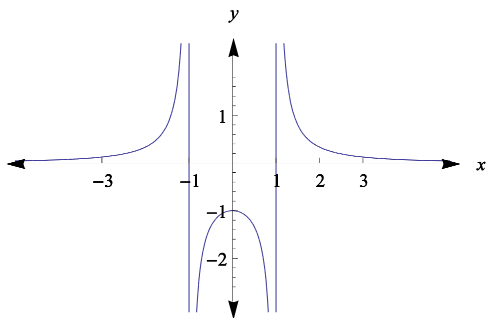
$n = 6$

3.11 Comments on Graphs of rational functions with denominator degree ≥ 2

Similar arguments apply to graphs of $f(x) = \dfrac{P(x)}{Q(x)}$ where P, Q are polynomial functions and the degree of $Q(x)$ is greater than or equal to 2.

We can apply the transformation rules to the basic graph of $y = 1/x$ to obtain a graph of $y = \dfrac{1}{x-1} + 2$ or the basic graph of $y = \dfrac{1}{x^2}$ to graph for $y = \dfrac{-1}{(x+1)^2}$, but in general we cannot use the rules to graph functions $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x) \geq 2$.

We saw for example in Section 2.3.5, that the graph of $y = \dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$ was,



In order to sketch a graph of the form $f(x) = \dfrac{P(x)}{Q(x)}$ we need to perform a similar analysis to that given in Example 3 of Section 2.3.5 to find the horizontal and vertical asymptotes.

We often also need the tools of Calculus to find the turning points.

Exercises 3C

Intercepts with the axes

In 1 to 5 show the x and y intercepts and test for symmetry.

1. Sketch the graph of $y = 3x + 3$
2. Sketch the graph of $y = x^2 + 2$
3. Sketch the graph of $y = 9 - x^2$
4. Sketch the graph of $y = \sqrt{x+4}$
5. Sketch the graph of $y = |4 - x|$
6. Sketch the graph of $y = 1 - x^2$ showing the x and y intercepts.
7. Sketch the graph of $y = x^2 + 2$ showing the x and y intercepts.
8. Sketch the graph of $y = 9 - x^2$ showing the x and y intercepts.
9. Sketch the graph of $y = \sqrt{x}$ showing the x and y intercepts.
10. Sketch the graph of $y = x^4$ showing the x and y intercepts.
11. Sketch the graph of $x^2 + y^2 = 4$ showing the x and y intercepts.
12. Graph $f(x) = 3x^2 - 12$
13. Graph $f(x) = x^2 - 8x + 15$

Symmetry

14. Test $x = y^4 - y^2$ for symmetry.
15. Test $y = x^3 + 10x$ for symmetry.

Summary of transformation rules

16. Match the equation in the first column with the instructions in the second column.

(a) $y=f(x-1)$	A. Translate left one unit
(b) $y=f(x)-1$	B. Reflect in the x-axis then translate left one unit
(c) $y=f(x)+1$	C. Translate right one unit
(d) $y=f(x+1)$	D. Reflect in the x-axis then translate up one unit
(e) $y=f(-x)+1$	E. Reflect in the x-axis then translate down one unit
(f) $y=-f(x)+1$	F. Translate down one unit
(g) $y=-f(x+1)$	G. Reflect in the x-axis, reflect in the y-axis, translate up one unit
(h) $y=-f(x)-1$	H. Translate left one unit, then reflect in the y-axis, then translate up one unit.
(i) $y=f(1-x)+1$	I. Translate up one unit.
(j) $y=-f(-x)+1$	J. Reflect in the y-axis, then translate up one unit.
(k) $y=f(2x)+1$	K. Compress horizontally by a factor of $1/2$, then translate up one unit

17. Match the equation in the first column with instruction/s in the second column.

(a) $y=f(x+2)+3$	A. Translate left 2 units, then translate down 3 units
(b) $y=f(x+3)+2$	B. Translate left 3 units, then translate up 2 units
(c) $y=f(x-2)+3$	C. Translate right 3 units then up 2 units
(d) $y=f(x-2)-3$	D. Translate left 3 units, then down 2 units
(e) $y=f(x+2)-3$	E. Translate right 3 units then down 2 units
(f) $y=f(x-3)+2$	F. Translate left 2 units, then stretch vertically by a factor of 3
(g) $y=f(x-3)-2$	G. Reflect in the x-axis, then translate right by 2 units
(h) $y=f(x+3)-2$	H. Reflect in the x-axis then translate left 2 units
(i) $y=-f(x+2)$	I. Translate left 2 units, then reflect in the y-axis
(j) $y=-f(x-2)$	J. Translate right 2 units then up 3 units
(k) $y=f(2-x)$	K. Translate left 2 units, then translate up 3 units
(l) $y=3f(x+2)$	L. Translate right 2 units and down 3 units.

Chapter 4

Exponentials and Logarithms

4.1 Definition of Exponential Functions

The function f defined by $f(x) = b^x, b > 0, b \neq 1, x \in \mathbb{R}$ is called the exponential function with base b.

4.2 Graphs of Exponential Functions

Since $b^0 = 1$, the graph of every exponential function passes through $(0,1)$.

Case 1: $b > 1$

When $x \to -\infty,\ f(x) \to b^{-\infty} = \dfrac{1}{b^{\infty}} = 0$, so the negative $x-axis$ is an asymptote.

When $x \to \infty,\ f(x) \to \infty$ very rapidly or exponentially.

The basic graph of $f(x) = b^x$ for $b > 1$ is shown in Figure 4.1.

Case 2: $b < 1$

The graph of $f(x) = b^x$ for $b < 1$ is shown in Figure 4.2. It is the reflection of $\left(\dfrac{1}{b}\right)^x$ in the $y-axis$ as we see in Example 2.

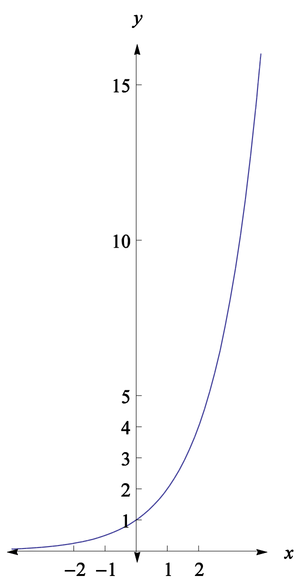


Figure 4.1 $b^x, b > 1$

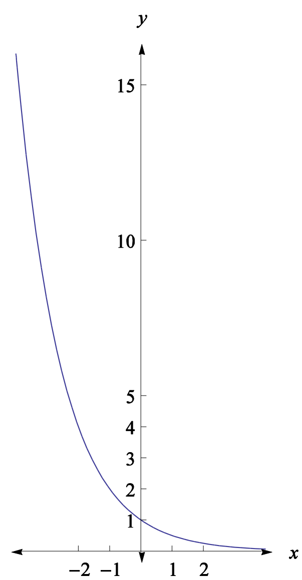


Figure 4.2 $b^x, b < 1$

Example 1

Sketch the graphs of $f(x) = 2^x,\ f(x) = 3^x$.

<u>Solution:</u> A simple table of values shows $3^x > 2^x$ for $x > 0$ and $2^x > 3^x$ for $x < 0$.

x	2^x	3^x
0	1	1
1	2	3
2	4	9
3	8	27
4	16	81
-1	0.5	0.33
-2	0.25	0.11

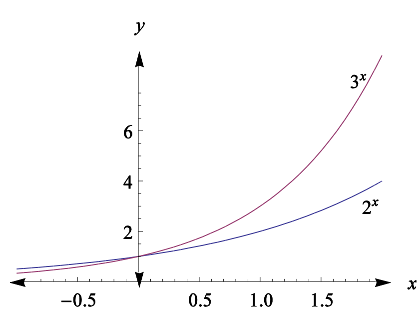


Figure 4.3 $y = 2^x,\ y = 3^x$

4.3 Transformations

Recall the graph of $y-k=af(b(x-h))$ is obtained from the basic graph of $y=f(x)$ as follows.
(a) If $a<0$, reflect in the $x-axis.$
(b) If $b<0$, reflect in the $y-axis.$
(c) The basic graph is translated so $(0,0)\to(h,k)$, the whole graph moves accordingly.
(d) If $|a|<1$, the basic graph is shrunk vertically by that factor, if $|a|>1$ the basic graph is stretched vertically by that factor.
(e) If $|b|>1$, the basic graph is shrunk horizontally by a factor of $b.$ If $|b|<1$, the basic graph is stretched horizontally by that factor.

We can use our translation, reflection, stretching and shrinking rules to draw extensions of the basic graphs of $f(x)=b^x$. Let's first consider an example of $b<1$.

Example 2

Sketch the graph of $f(x)=\left(\frac{1}{2}\right)^x$

<u>Solution</u>: $f(x)=\left(\frac{1}{2}\right)^x=2^{-x}$, so compared with our basic graph of $f(x)=2^x$, we want to sketch $f(-x)=2^{-x}$. This is simply the reflection of $f(x)=2^x$ in the $y-axis.$

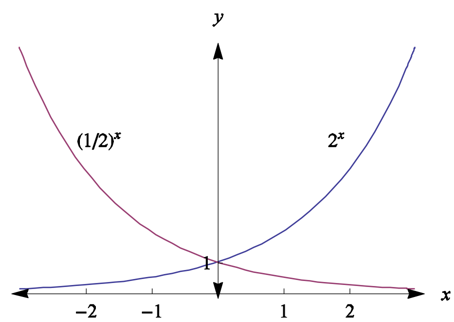


Figure 4.4 $y=2^x$, $y=\left(\frac{1}{2}\right)^x$

Example 3

Sketch (a) $f(x)=2^{x-1}$, (b) $f(x)=-2^x$

<u>Solution:</u>

(a) This is a translation of $f(x)=2^x$ so that $(0,0)\to(1,0)$. This also takes $(0,1)\to(1,1)$. Figure 4.5.

(b) This is a reflection of the basic graph $f(x)=2^x$ in the $x-axis.$ Figure 4.6.

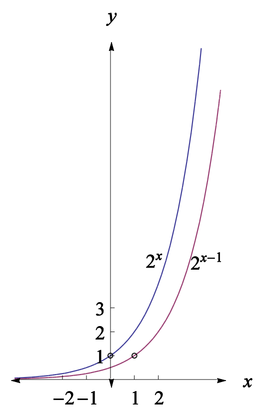


Figure 4.5 $f(x)=2^{x-1}$

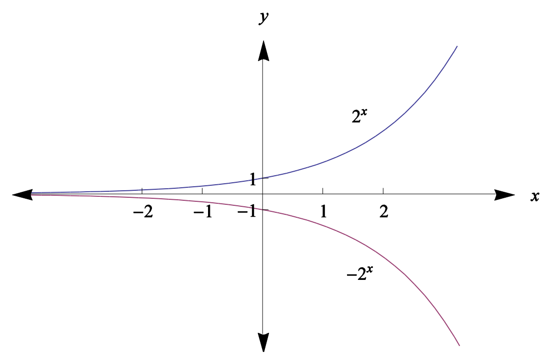


Figure 4.6 $f(x)=-2^x$

Example 4

Sketch $f(x)=2^{x+1}-2$

<u>Solution:</u> Write this as $y+2=2^{x+1}$. The basic graph $f(x)=2^x$ is translated so that $(0,0)\to(-1,-2)$. This means $(0,1)\to(-1,-1)$ and $y=0\to y=-2$ which is the new asymptote. The new graph passes through the origin since $f(x)=2^{x+1}-2=0\Rightarrow x=0$.

See Figure 4.7.

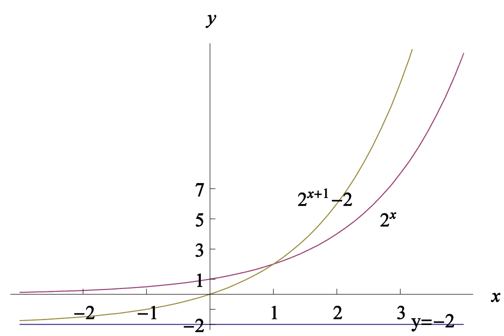


Figure 4.7 $f(x)=2^{x+1}-2$

4.4 The natural base e

The exponential function that occurs naturally in many settings is

$$y=f(x)=e^x\text{, where } e=2.718281828.....$$

This is called the natural exponential function. Since $2<e<3$, its graph is situated as shown in Figure 4.8.

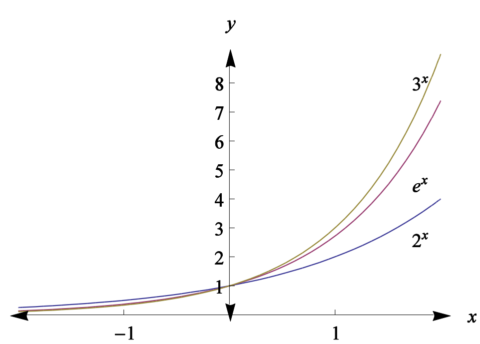


Figure 4.8

4.5 Some notes on e

If you set up a table of values for $x-$ values close to zero such as:

x	2^x	3^x
0	1	1
0.1	1.07	1.12

and use $m=\dfrac{y_2-y_1}{x_2-x_1}$ for these two line segments from $x=0$ to $x=0.1$ that are very similar to the respective tangents at $x=0,$ we can see that the gradients or slopes of the tangents to $f(x)=2^x, 3^x$ are about $0.7, 1.2$ respectively. Obviously the gradient of $f(x)=e^x$ at $x=0$ is therefore between 0.7 and $1.2.$ One way of defining $f(x)=e^x$ is that it is the exponential function whose gradient at $x=0$ is exactly 1.

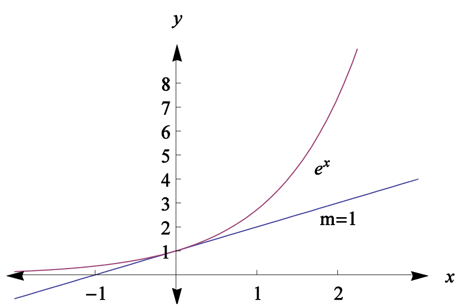


Figure 4.9 Tangent of $f(x)=e^x$ at $(0,1)$

In Calculus, we will prove $f(x)=e^x$ can be expressed as an infinite polynomial $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+...+\dfrac{x^n}{n!}+....$ By putting $x=1$, we can easily calculate e to as many decimal places as we like:

$$e=2.718281828...$$

The pattern of decimals does not repeat, so e is an irrational number. (You know from your algebra course that any repeating decimal can be expressed as a geometric series with its common ratio $r<1$, so, using $S_\infty=\dfrac{a}{1-r}$, it is a rational number.)

In fact, e is more than an irrational number. Irrationals such as $\sqrt{2}$ are the solutions to a polynomial equation with integer coefficients, in this case, $x^2-2=0$. This is not true for e. It is called a transcendental number. The number π is also transcendental.

An interesting fact is that the transcendentals e and π as well as the complex number $i=\sqrt{-1}$ are related by an intriguingly simple formula we can also prove in Calculus, namely, $e^{\pi i}=-1$.

4.6 Solving exponential equations

As we will also see in Calculus, the basic exponential rules listed below that we have from algebra apply to any positive real number bases and any real number exponents and not just to integers. We can state the extended rules as follows:

<u>Rules for operations on exponential functions.</u>

If a and b are positive real numbers and m, n are any real numbers, then:

1. $b^m \bullet b^n = b^{m+n}$
2. $b^m \div b^n = b^{m-n}$
3. $(b^m)^n = b^{mn}$
4. $(ab)^n = a^n \bullet b^n$
5. $\dfrac{b^m}{b^n} = b^{m-n} \quad or \quad \dfrac{1}{b^{n-m}}$
6. $\dfrac{1}{b^m} = b^{-m}, \ \dfrac{1}{b^{-m}} = b^m$
7. $b^0 = 1$
8. $(-1)^m = 1$ if m is even, and -1 if m is odd.

Example 5

Simplify (a) $2^3 \bullet 2^2$, (b) $3^{\sqrt{2}} \bullet 3^{\sqrt{5}+1}$

<u>Solution</u>: (a) $2^3 \bullet 2^2 = 2^5$, (b) $3^{\sqrt{2}} \bullet 3^{\sqrt{5}+1} = 3^{\sqrt{2}+\sqrt{5}+1}$

We can use these rules to solve exponential equations.

Example 6

Solve $3^{2x} = \dfrac{1}{9}$

<u>Solution:</u> $3^{2x} = \dfrac{1}{9} = \dfrac{1}{3^2} = 3^{-2}$. Then, setting the exponents equal, $2x = -2 \Rightarrow x = -1$.

Example 7

Solve $8^x = 32$

<u>Solution:</u> $8^x = 32 \Leftrightarrow (2^3)^x = 2^5 \Leftrightarrow 2^{3x} = 2^5 \Rightarrow 3x = 5 \Rightarrow x = \dfrac{5}{3}$

Example 8

Solve $(3^{x+1})(9^{x-3}) = 81$

<u>Solution</u>: $(3^{x+1})(9^{x-3}) = 81 \Leftrightarrow (3^{x+1})((3^2)^{x-3}) = 3^4 \Leftrightarrow 3^{x+1} \bullet 3^{2x-6} = 3^4 \Leftrightarrow 3^{3x-5} = 3^4 \Rightarrow x = 3$

4.7 Applications

4.7.1 Compound Interest

If a sum of money, called the principal P, is invested at $r\%$ and the interest is added annually or compounded annually, the amount A resulting,

After 1 year is $\quad A = P + rP = P(1+r)^1$

After 2 years is $\quad A = P(1+r)^1 + rP(1+r) = P(1+r)^2$

…

After t years is $\quad A = P(1+r)^t$

If the money invested is compounded more than once a year, say n times a year (e.g., monthly is $n = 12$, weekly is $n = 52$), we can similarly show that after t years,

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

An important formula, proved in Calculus, is that if the money is compounded continuously $(n \to \infty)$, then,

$$A = Pe^{rt}$$

Example 9

A total of \$6000 is invested in an interest account bearing 9% annually. Find the amount resulting after 5 years if the interest is compounded (a) monthly, (b) continuously.

<u>Solution:</u>

(a) $A = P\left(1 + \frac{r}{n}\right)^{nt} \Leftrightarrow A = 6000\left(1 + \frac{0.09}{12}\right)^{12\bullet 5} \approx \$9,394$

(b) $A = Pe^{rt} \Leftrightarrow A = 6000e^{0.09\bullet 5} \approx \$9,410$

4.7.2 Exponential Growth and Decay in general

By similar arguments,

$$Q(t) = Q_0 e^{kt}$$

is a mathematical model for numerous growth and decay applications. Here $Q(t)$ is the quantity of a given substance at time t, Q_0 is the initial amount of the substance when $t = 0$, and k is a constant that depends on the particular application. If $k > 0$ we have exponential growth and if $k < 0$ we have exponential decay.

Example 10
Suppose the number of bacteria $Q(t)$ present in a culture after t minutes is given by $Q(t)=Q_0e^{0.05\,t}$ where Q_0 is the initial population. If 5000 bacteria are present after 20 minutes, how many were present initially?
Solution: We have $Q(20)=5000=Q_0e^{0.05(20)}=Q_0e\Rightarrow Q_0=\dfrac{5000}{e}=1839$

Example 11
The number of grams of a radioactive substance present after t seconds, is given by $Q(t)=200e^{-0.3\,t}$. (a) How many grams remain after 7 seconds? (b) What is the half-life of the substance?
Solution:
(a) $Q(7)=200e^{-0.3\times 7}=24.5\,g$
(b) The half-life means the time when just half the original material remains, that is the value of t when $Q(t)=\dfrac{Q_0}{2}$. Substituting, $\dfrac{Q_0}{2}=Q_0e^{-0.3\,t}\Leftrightarrow e^{-0.3\,t}=0.5$.
We need logarithms to solve this equation! See Example 26 below for the answer.

Exercises 4A

1. Graph $f(x)=2^{x+2}$
2. Graph $f(x)=\left(\dfrac{1}{3}\right)^x$
3. Graph $f(x)=2^x+1$
4. Graph $f(x)=2^{-x-2}$
5. Solve $3^{2x}=27$
6. Solve $\left(\dfrac{1}{2}\right)^x=\dfrac{1}{128}$
7. Solve $6^{3x-1}=36$
8. Solve $27^{4x}=9^{x+1}$
9. Solve $10^x=0.1$
10. Solve $2^x=64$
11. Solve $3^{-x}=\dfrac{1}{243}$
12. Solve $16^x=64$
13. Solve $9^{4x-2}=\dfrac{1}{81}$
14. Solve $(27)(3^x)=9^x$
15. Solve $(2^{x+1})(2^x)=64$

In 16 and 17 use the formula $A=P\left(1+\dfrac{r}{n}\right)^{nt}$ to find the total amount of money accumulated at the end of each indicated time period:

16. \$200 for 6 years at 6% compounded annually.
17. \$800 for 9 years at 5% compounded quarterly.

In 18 and 19 use the formula $A=Pe^{rt}$ to find the amount of money accumulated at the end of the time period compounded continuously:

18. \$400 for 5 years at 7%
19. \$500 for 7 years at 6%

20. Suppose a certain radioactive substance has a half-life of 20 years. If there are presently 2500 milligrams of the substance, how much, to the nearest milligram, will remain after (a) 40 years, (b) 50 years?
21. The half-life of radium is approximately 1600 years. If the present amount of radium is a certain location is 500 grams, how much will remain after 800 years? Express your answer to the nearest gram.
22. Suppose in a certain culture, the equation $Q(t) = 1000e^{0.4t}$ expresses the number of bacteria present as a function of time t, where t is in hours. How many bacteria are present at the end of (a) 2 hours, (b) 3 hours, (c) 5 hours?

=====================================_

4.8 Logarithmic Functions

Definition: The inverse of the exponential function $y = f(x) = b^x$ with base b is called the **logarithmic function with base** b, written $y = f^{-1}(x) = \log_b x$.

We obtain the inverse function of $y = b^x$ as usual by interchanging the variables to get $x = b^y$. So we have the equivalence

$$y = \log_b x \Leftrightarrow x = b^y$$

The first expression is in logarithmic form, the second is in exponential form. Note, that a logarithm therefore is an exponent. y is the power or exponent to which the base b must be raised to get x.

Example 12

Write the equivalent statement of the following:

(a) $2^3 = 8$, (b) $10^2 = 100$, (c) $\log_6 216 = 3$

Solution

$(a)\ 2^3 = 8 \Leftrightarrow \log_2 8 = 3$

$(b)\ 10^2 = 100 \Leftrightarrow \log_{10} 100 = 2$

$(c)\ 216 = 6^3$

Example13

Evaluate (a) $\log_{10} 0.0001$ (b) $\log_2 32$

Solution:

(a) $y = \log_{10} 0.0001 \Leftrightarrow 0.0001 = 10^y \Leftrightarrow 10^{-4} = 10^y \Rightarrow y = -4$

(b) $y = \log_2 32 \Leftrightarrow 32 = 2^y \Leftrightarrow 2^5 = 2^y \Rightarrow y = 5$

Example 14

Solve $\log_8 x = \dfrac{2}{3}$

Solution: $\log_8 x = \dfrac{2}{3} \Leftrightarrow x = 8^{\frac{2}{3}} \Leftrightarrow x = (2^3)^{\frac{2}{3}} \Leftrightarrow x = 2^2 \Leftrightarrow x = 4$

4.9 Graphs of logarithmic functions

Since the logarithmic function is the inverse of the exponential function, the graph of $y = \log_b x$ is obtained by reflecting the graph of $y = b^x$ in the line $y = x$.

Notes: Every graph of $y = \log_b x$

(a) passes though $(1,0)$ since $(0,1) \to (1,0)$

(b) approaches the negative $y-axis$ asymptotically

(c) grows very slowly as $x \to \infty$ (for example, $\log_{10} 10 = 1, \log_{10} 1000 = 3$)

(d) The domain of $y = \log_b x$ is $(0, \infty)$ or $x > 0$ only.

Note: The logarithm of negative numbers and zero is undefined. You cannot evaluate the logarithm of a negative number or zero!

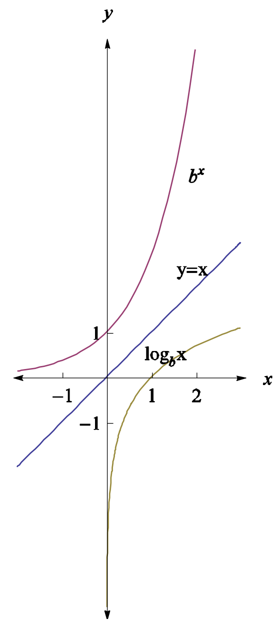


Figure 4.10 $y = \log_b x$

Using the graph of $y = \log_b x$ as the basic graph, we can use our rules for translations, reflections, stretchings and shrinkings to graph extended log functions.

Example 15

Sketch (a) $f(x) = \log_2(x-1)$, (b) $f(x) = 2 + \log_2 x$, (c) $f(x) = \log_2(x+1) - 3$, (d) $f(x) = 2 - \log_2 x$

Solution:

(a) This is a translation of $y = \log_2 x$ so that $(0,0) \to (1,0)$ and the asymptote shifts from the $y-axis$ or $x = 0$ to the vertical line $x = 1$. See Figure 4.11.

(b) Rewritten as $y - 2 = \log_2 x$, this is a translation of $y = \log_2 x$ so that $(0,0) \to (0,2)$ and $(1,0) \to (1,2)$. See Figure 4.12.

(c) Rewritten as $y+3=\log_2(x+1)$, this is a translation of $y=\log_2 x$ so that $(0,0)\to(-1,3)$, shifting the vertical asymptote to the line $x=-1$ and $(1,0)\to(0,3)$. The $y-axis$ intercept is -3. See Figure 4.13.

(d) This is a translation of $y=\log_2 x$ so that $(0,0)\to(0,-2)$ and $(1,0)\to(1,-2)$ and then a reflection in the $x-axis$ making $(1,0)\to(1,+2)$ See Figure 4.14.

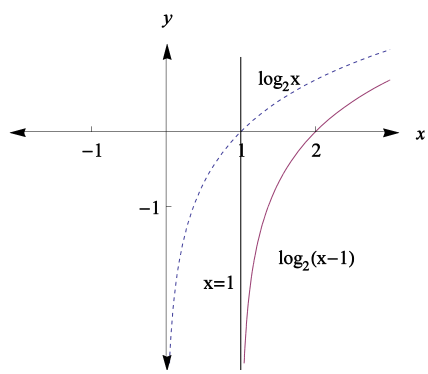


Figure 4.11 $f(x)=\log_2(x-1)$

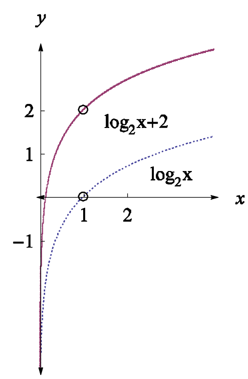


Figure 4.12 $f(x)=2+\log_2 x$

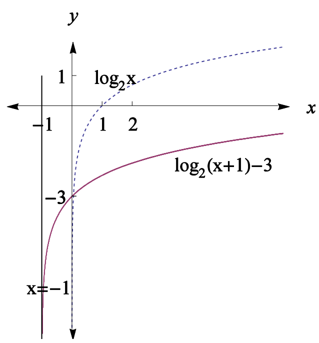


Figure 4.13 $y+3=\log_2(x+1)$

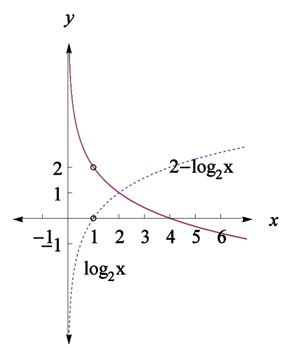


Figure 4.14 $f(x)=2-\log_2 x$

4.10 The natural logarithmic function

$y=\log_e x$ written as $y=\ln x$ is the inverse function of the natural exponential function $y=e^x$. Their graphs are shown in Figure 4.15.

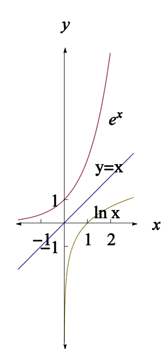


Figure 4.15 $y=\ln x$

You can evaluate the natural logarithm function by using a calculator and the $\boxed{\ln}$ key.

Example 16
Use your calculator to find (a) $\ln 2$, (b) $\ln 0.3$
Solutions:
$\ln 2 = 0.6931472$, $\quad \ln 0.3 = -1.2039728$

Exercises 4B

1. Write $\log_4 64 = 3$ in exponential form.
2. Write $\log_{10} 10,000 = 4$ in exponential form.
3. Write $\log_{10} 0.001 = -3$ in exponential form.
4. Write $2^4 = 16$ in logarithmic form.
5. Write $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$ in logarithmic form.
6. Evaluate $\log_2 16$
7. Evaluate $\log_7 \sqrt{7}$
8. Evaluate $10^{\log_{10} 5}$
9. Evaluate $\log_5(\log_2 32)$
10. Evaluate $\log_3 81$
11. Evaluate $\log_4\left(\frac{1}{64}\right)$
12. Evaluate $3^{\log_3 3}$
13. Solve $\log_8 x = \frac{4}{3}$
14. Solve $\log_9 x = \frac{3}{2}$
15. Solve $\log_{27} x = \frac{1}{3}$

In 12, 13 and 14 use the basic graph $f(x) = \log_2 x$.

16. Graph $f(x) = 3 + \log_2 x$
17. Graph $f(x) = -2 + \log_2 x$
18. Graph $f(x) = \log_2(x-2)$

19. Use your calculator to find $\ln 5$
20. Use your calculator to solve $\ln x = 0.4721$

==

4.11 Properties of logarithms

Most of the properties of logarithms are a direct consequence of the definition

$$y = \log_b x \Leftrightarrow x = b^y$$

and the inverse function relationship between $y = b^x$ and $y = \log_b x$. The others can be proved by using the exponential laws.

Properties of logarithms

1. $\log_b b^x = x,\ b^{\log_b x} = x$
2. $\log_b b = 1$
3. $\log_b 1 = 0$
4. $\log_b xy = \log_b x + \log_b y$
5. $\log_b \dfrac{x}{y} = \log_b x - \log_b y$
6. $\log_b x^r = r\log_b x$

In particular,

a. $\ln \mathrm{e}^x = x,\ e^{\ln x} = x$

b. $\ln e = 1$

c. $\ln 1 = 0$

Proofs

1. This is just $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
2. $y = \log_b b \Leftrightarrow b^y = b^1 \Rightarrow y = 1$
3. $y = \log_b 1 \Leftrightarrow b^y = 1 = b^0 \Rightarrow y = 0$
4. Let $m = \log_b x,\ n = \log_b y$.

   Then $x = b^m,\ y = b^n \Rightarrow xy = b^m \bullet b^n = b^{m+n} \Rightarrow \log_b xy = m + n = \log_b x + \log_b y$
5. Same idea as proof of 4. $\dfrac{x}{y} = b^{m-n}$, etc.
6. $m = \log_b x \Leftrightarrow x = b^m \Rightarrow x^r = (b^m)^r = b^{rm} \Leftrightarrow \log_b x^r = rm = r\log_b x$

Example 17

Simplify $(a)\ \ln e,\ (b)\ e^{\ln 5},\ (c)\ \dfrac{\ln 1}{3},\ (d)\ \log_5 \sqrt[3]{5},\ (e)\ \ln e^6 - \ln e^2$

<u>Solutions:</u>

$$(a)\ln e = 1\ \ (b)\ e^{\ln 5} = 5\ \ (c)\frac{\ln 1}{3} = \frac{0}{3} = 0$$

$$(d)\log_5 \sqrt[3]{5} = \log_5 5^{\frac{1}{3}} = \frac{1}{3}\log_5 5 = \frac{1}{3}$$

$$(e)\ln e^6 - \ln e^2 = 6\ln e - 2\ln e = 6 - 2 = 4$$

Example 18

Expand $\log_3 5x^3y^2$

Solution: $\log_3 5x^3y^2 = \log_3 5 + \log_3 x^3 + \log_3 y^2 = \log_3 5 + 3\log_3 x + 2\log_3 y$

Example 19

Expand $\ln\dfrac{\sqrt{3x-7}}{4}$

Solution: $\ln\dfrac{\sqrt{3x-7}}{4} = \ln\sqrt{3x-7} - \ln 4 = \ln(3x-7)^{\frac{1}{2}} - \ln 4 = \dfrac{1}{2}\ln(3x-7) - \ln 4$

Example 20

Condense $\dfrac{1}{2}\log_{10} x + 3\log_{10}(x+1)$

Solution: $\dfrac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{\frac{1}{2}} + \log_{10}(x+1)^3 = \log_{10}[x^{\frac{1}{2}}(x+1)^3]$

Example 21

Condense $\dfrac{1}{5}[2\ln(x+2) - \ln x]$

Solution: $\dfrac{1}{5}[2\ln(x+2) - \ln x] = \dfrac{1}{5}[\ln(x+2)^2 - \ln x] = \dfrac{1}{5}\ln\left(\dfrac{(x+2)^2}{x}\right) = \ln\left(\dfrac{(x+2)^{\frac{2}{5}}}{x^{\frac{1}{5}}}\right)$

Example 22

If $\log_5 36 = 2.2266$, $\log_5 4 = 0.8614$, evaluate $\log_5 9$

Solution: $\log_5 9 = \log_5 \dfrac{36}{4} = \log_5 36 - \log_5 4 = 2.2266 - 0.8614 = 1.3652$

Example 23

Evaluate $\log_2 22^{\frac{1}{3}}$ given $\log_2 22 = 4.4598$

Solution: $\log_2 22^{\frac{1}{3}} = \dfrac{1}{3}\log_2 22 = \dfrac{4.4598}{3} = 1.4866$

Example 24

Solve $\log_5(x+4) - \log_5 x = 2$

Solution:

$$\log_5(x+4) - \log_5 x = 2 \Leftrightarrow \log_5 \frac{x+4}{x} = 2 \Leftrightarrow \frac{x+4}{x} = 5^2 = 25 \Leftrightarrow 25x = x+4 \Rightarrow x = \frac{1}{6}$$

Example 25

Solve $\log_{10} x + \log_{10}(x+9) = 1$

<u>Solution:</u>

$\log_{10} x + \log_{10}(x+9) = 1 \Leftrightarrow \log_{10} x(x+9) = 1 \Leftrightarrow x(x+9) = 10^1$

$\Leftrightarrow x^2 + 9x - 10 = 0 \Leftrightarrow (x+10)(x-1) = 0 \Rightarrow x = -10, 1$

Since we only define $\log x$ for $x > 0$,we discount the value -10. The solution is $x = 1$.

Example 26 (completion of Example 11)

Solve $e^{-0.3t} = 0.5$

<u>Solution:</u> $e^{-0.3t} = 0.5 \Leftrightarrow \ln(e^{-0.3t}) = \ln(0.5) \Leftrightarrow -0.3t = -0.693 \Rightarrow t = 2.31$

Example 27

Evaluate $2^{3\log_2 4}$

<u>Solution:</u> $2^{3\log_2 4} = 2^{\log_2 4^3} = 4^3 = 64$

Example 28

Solve $4^x = 35$

<u>Solution:</u>

$$\ln 4^x = \ln 35 \Rightarrow x \ln 4 = \ln 35 \Rightarrow x = \frac{\ln 35}{\ln 4} = 2.56$$

4.12 Ancient History

It is hard to believe that until about 30 years ago, logarithms to base 10 were the only way to evaluate complicated mathematical expressions such as

$$x = \frac{2.7^3 \times \sqrt{10.67}}{11.31}$$

apart from doing it all by hand.

Anyone needing to evaluate such expressions had a Table of Logarithms. One wrote $\log x = 3\log 2.7 + \frac{1}{2}\log 10.67 - \log 11.31$ and "looked up" the log tables to find the value of the right side (it is 0.75474752…). Then to find x such that $\log x = 0.75474752$ one "looked up" a Table of Anti-logarithms" to find $x = 5.685226242$. Mostly the Tables would only give answers to four decimal places.

One could also use a Slide Rule which was based on the conversion of numbers to logs, but the precision was then even smaller. Then came the advent of the personal computer and handheld calculators. As they say, the rest is history!

Logarithms to base 10 were used so commonly, they were called common logarithms and it was understood that if the base were omitted, then always $\log x = \log_{10} x$. The $\boxed{\text{log}}$ key on your calculator, which you probably never use, and which is just a nod to history, means $\log_{10}$. Just for fun, type 100 into your calculator and press the $\boxed{\log}$ key. It gives the answer 2, as expected since

$$y = \log_{10} 100 \Leftrightarrow 100 = 10^y \Leftrightarrow 10^2 = 10^y \Rightarrow y = 2$$

But now, common logarithms are consigned to the mathematical scrapheap. They are ancient history!

4.13 Change of Base

Your calculator has $\boxed{\log}$ and $\boxed{\ln}$ keys for $\log_{10}$ and $\log_e$ respectively. You can use the change of base formula

$$\log_b c = \frac{\log_a c}{\log_a b}$$

to find the logarithms of numbers to other bases.

<u>*Proof*</u>

Let $y = \log_b c \Leftrightarrow b^y = c$.

$$\Rightarrow \log_a b^y = \log_a c$$

$$\Rightarrow y \log_a b = \log_a c$$

$$\Rightarrow y = \frac{\log_a c}{\log_a b}$$

Example 29

Use the $\boxed{\log}$ key on your calculator to find $\log_2 11$

<u>Solution</u> $\log_2 11 = \dfrac{\log 11}{\log 2} = \dfrac{1.04149}{0.30103} = 3.4594$

Exercises 4C

Find the exact value of 1. to 4. without using a calculator.

1. $\log_5 \dfrac{1}{125}$
2. $\log_6 \sqrt[3]{6}$
3. $\log_3 81^{-3}$
4. $2\ln e^6 - \ln e^5$

In 5. to 9. condense the expression into a logarithm of a single quantity.

5. $\log_5 8 - \log_5 t$
6. $2\log_2 x + 4\log_2 y$
7. $-4\log_6 2x$
8. $2\ln 8 + 5\ln(z-4)$
9. $2[3\ln x - \ln(x+1) - \ln(x-1)]$

Use the properties of logarithms to express the expressions in 10 to 13 as a sum, difference, and/or constant multiple of logarithms.

10. $\log_3 10z$
11. $\log_{10} \dfrac{y}{2}$
12. $\ln\left(\dfrac{x^2-1}{x^3}\right),\ x > 1$
13. $\ln\sqrt{\dfrac{x^2}{y^3}}$

Simplify 14 and15 using the properties of logarithms

14. $\log_2(4^2 \bullet 3^2)$
15. $\log_{10} \dfrac{9}{300}$

16. Solve $\ln(2+x) = 1$
17. Solve $\log_{10}(x-4) = 3$
18. Solve $\log_3(2-x) = 3$
19. Solve $\log_2(x^2 - x - 2) = 2$
20. Solve $2\log_{10} x = \log_{10} 2 + \log_{10}(3x-4)$
21. Solve $\log_5 x + \log_5(x+1) = \log_5 20$
22. Solve $\log_{10} x + \log_{10}(x+21) = 2$
23. Solve $\log_{10}(3x-1) = 1 + \log_{10}(5x-2)$
24. Solve $\log_{10}(x+1) = \log_{10} 3 - \log_{10}(2x-1)$

Solve the following equations for x:

25. Solve $3^x = 13$
26. Solve $2^x + 7 = 50$
27. Solve $e^{x-2} = 13.1$
28. Solve $5^{2x+1} = 7^{x+3}$
29. $2^{3x} = 34$
30. $3^{\frac{x}{14}} = 0.1$
31. $e^{3-5x} = 16$

32. $\left(\frac{1}{4}\right)^x = 75$

Evaluate by using the change of base formula.

33. $\log_7 4$

34. $\log_{\frac{1}{4}} 5$

35. $\log_{20} 0.25$