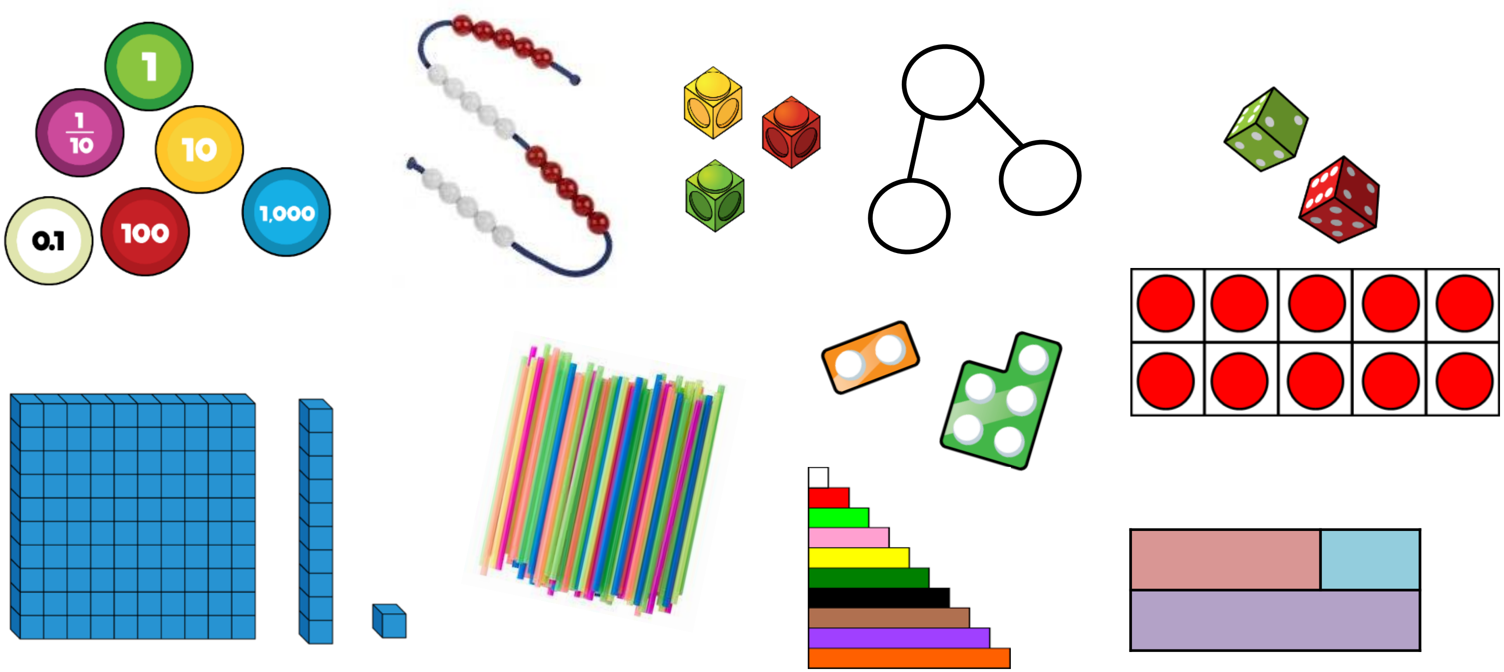
Discuss…

Which of these resources do you currently use in your classroom?



Concrete, Pictorial and Abstract Methods

Course Lead

White Rose Maths

@WhiteRoseMaths

Importance of CPA

In his research on the cognitive development of children (1966), Jerome Bruner proposed three modes of representation:

- Enactive representation (concrete)
- Iconic representation (pictorial)
- Symbolic representation (abstract)

'*From concrete manipulatives and experiences, students are **guided to uncover abstract mathematical concepts or results**... The role of the teacher is that of a **facilitator** who guides students through the concrete, pictorial and abstract levels of understanding by providing **appropriate scaffolding and feedback**.*'

Ministry of Education (2012)

Content

- Same concept, different representations
- Making connections
- Understanding the problem
- Proving why

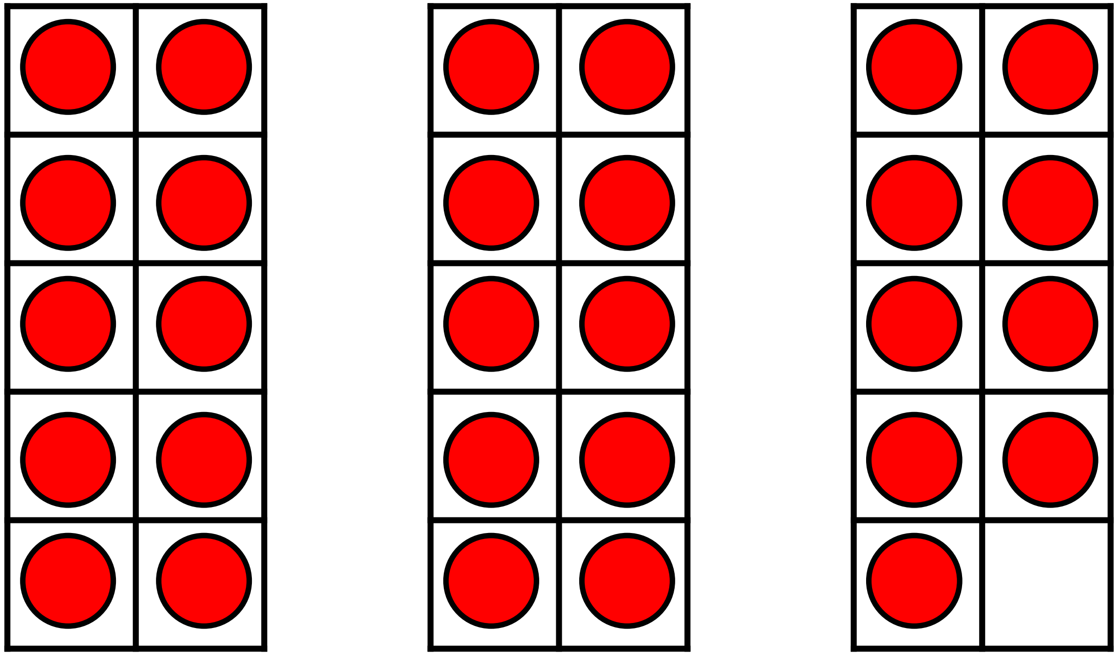
Same concept, different representations

‘A mathematical concept or skill has been mastered when, through exploration, clarification, practice and application over time, a person can represent it in multiple ways, has the mathematical language to be able to communicate related ideas, and can think mathematically with the concept so that they can independently apply it to a totally new problem in an unfamiliar situation.’

Drury, H. (2015)

Same concept, different representations

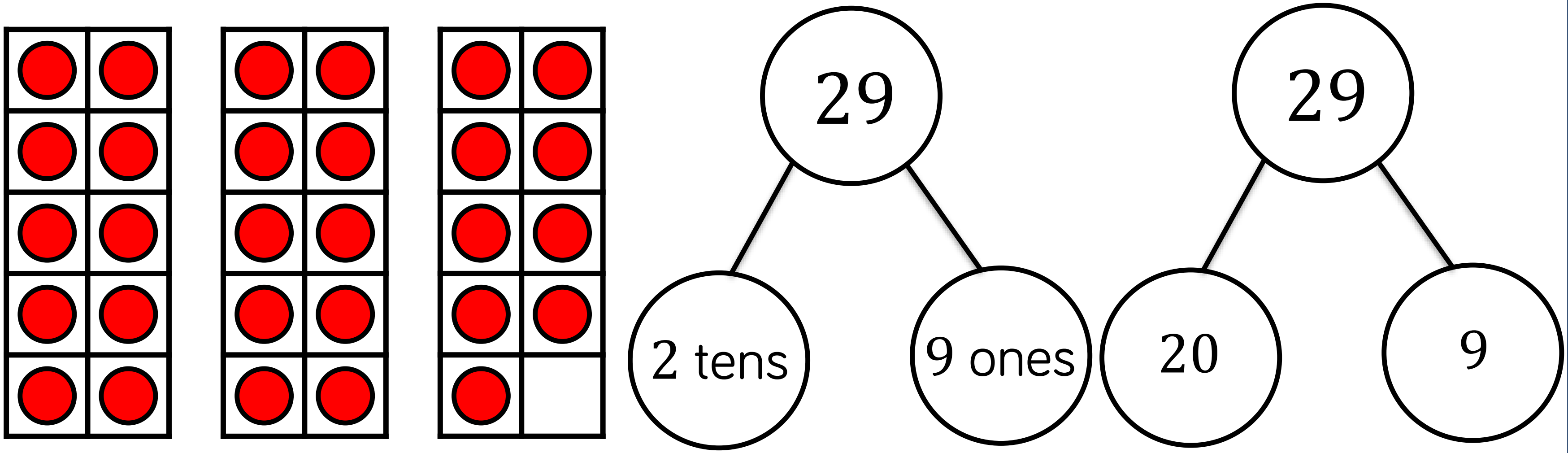
How can we represent the number 29?



What does this tell us about the number 29?

Same concept, different representations

How can we represent the number 29?



$29 = 29$ ones

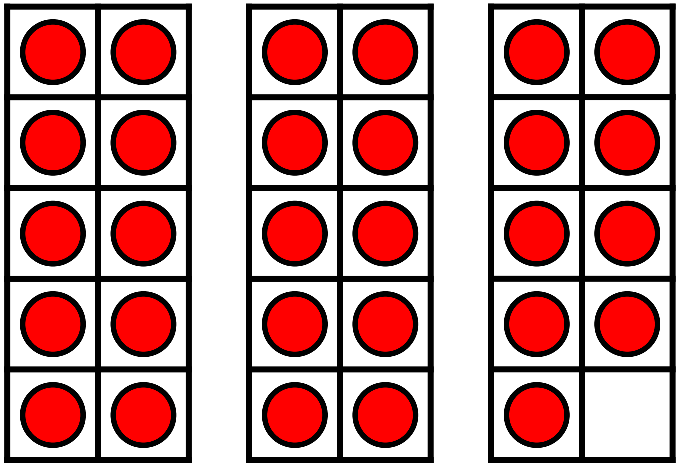
$29 = 2$ tens $+ 9$ ones

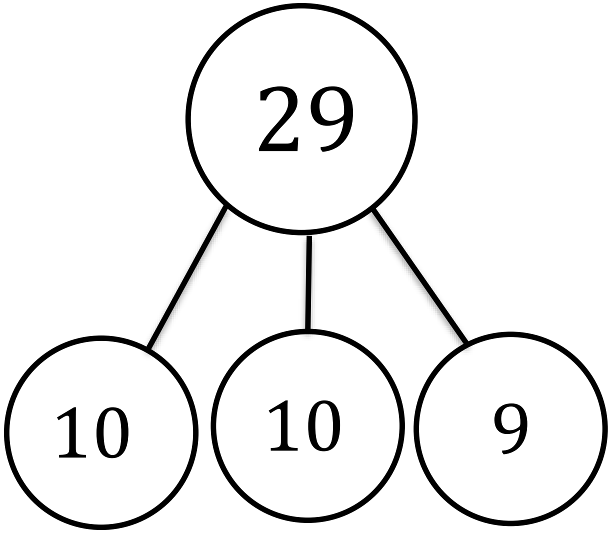
$29 = 20 + 9$

$29 = 10 + 10 + 9$

Same concept, different representations

How can we represent the number 29?





$29 = 29$ ones

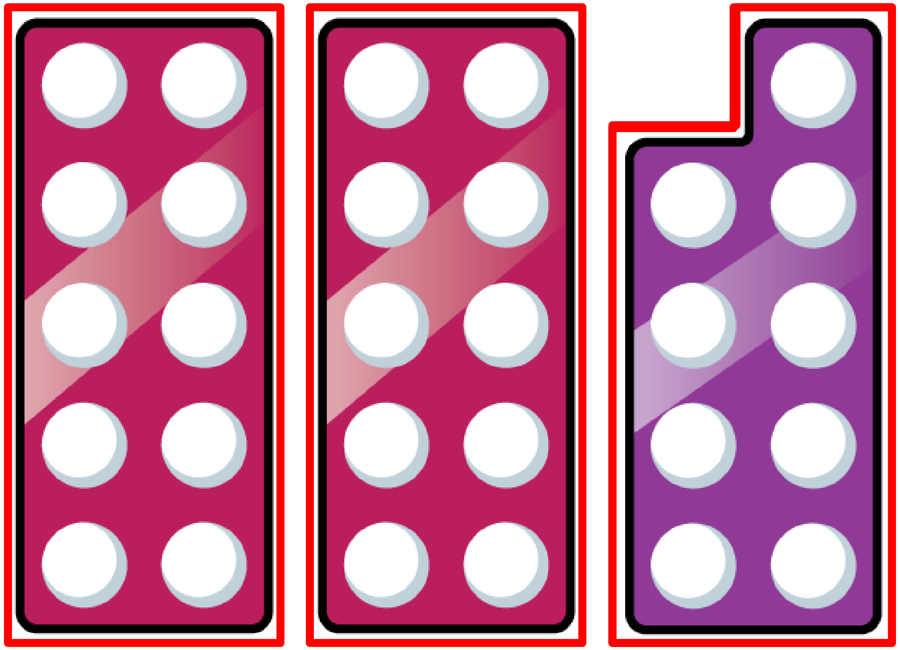
$29 = 2$ tens $+ 9$ ones

$29 = 20 + 9$

$29 = 10 + 10 + 9$

Same concept, different representations

How can we represent the number 29?



$29 = 10 + 10 + 9$

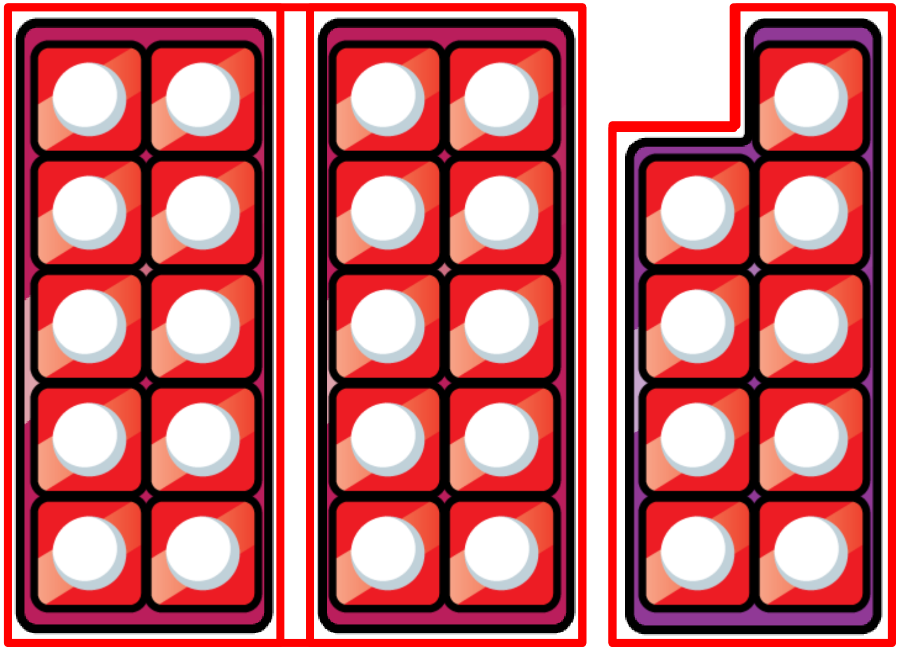
$29 = 2 \text{ tens} + 9 \text{ ones}$

$29 = 20 + 9$

What does this tell us about the number 29?

Same concept, different representations

How can we represent the number 29?



$29 = 10 + 10 + 9$

$29 = 2$ tens $+ 9$ ones

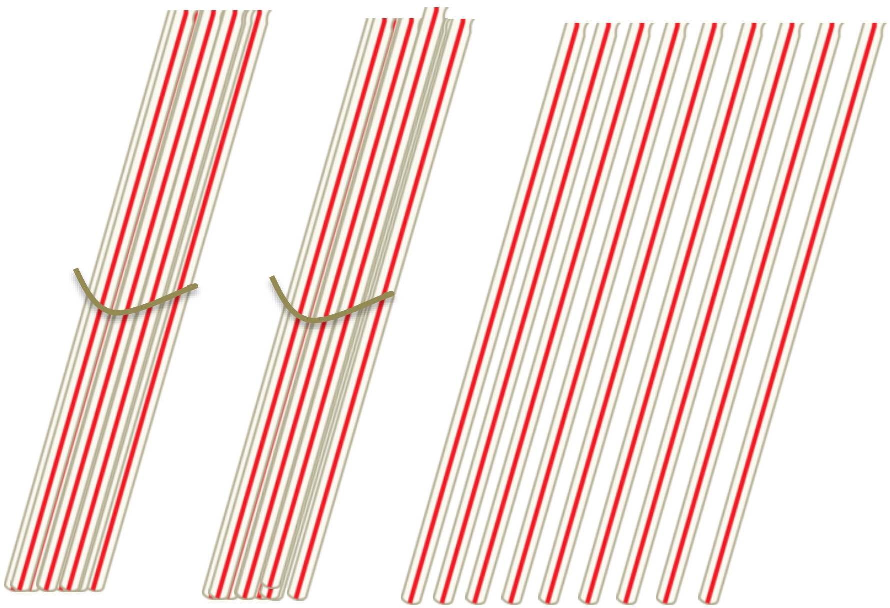
$29 = 20 + 9$

$29 = 29$ ones

What does this tell us about the number 29?

Same concept, different representations

How can we represent the number 29?



$$29 = 10 + 10 + 9$$

$$29 = 2 \text{ tens} + 9 \text{ ones}$$

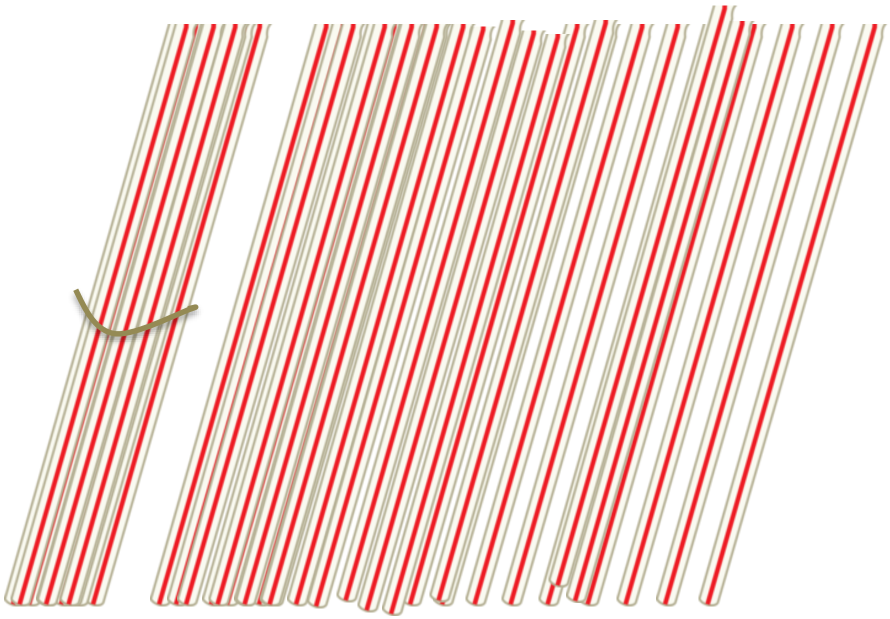
$$29 = 20 + 9$$

$$29 = 29 \text{ ones}$$

What does this tell us about the number 29?

Same concept, different representations

How can we represent the number 29?



$29 = 2$ tens $+ 9$ ones

$29 = 20 + 9$

$29 = 10 + 10 + 9$

$29 = 29$ ones

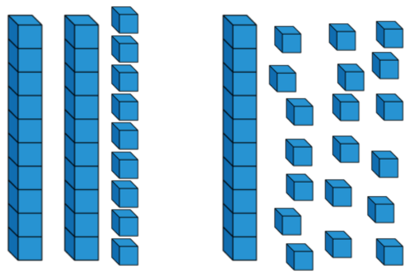
$29 = 1$ ten $+ 19$ ones

$29 = 10 + 19$

How could you represent this in a part whole model?

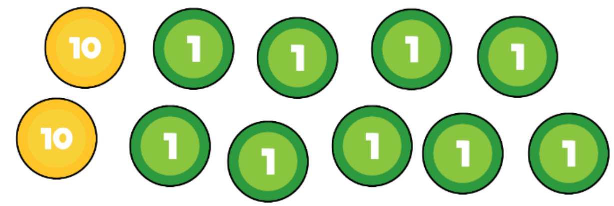
Same concept, different representations

Base 10

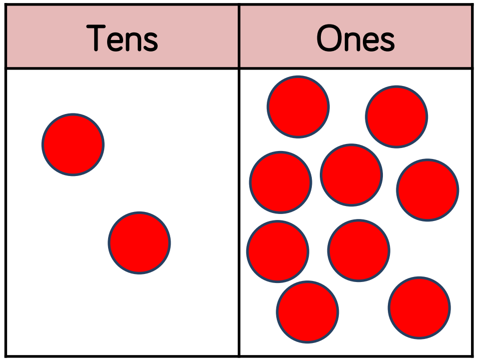


29

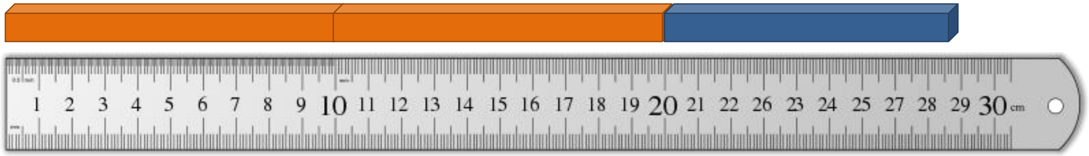
Place value counters



Place value grid



Cuisenaire rods



‘Students may require concrete materials to build meaning initially, but they must reflect on their actions with manipulatives to do so. They need teachers who can reflect on their students’ representations for mathematical ideas and help them develop increasing sophisticated and mathematical representations.’

Clements, D. (1999)

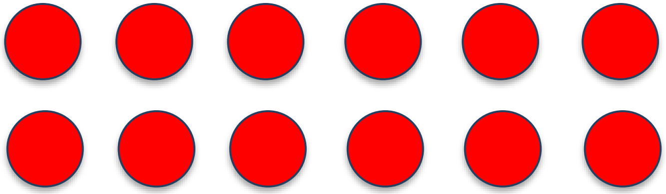
Making connections

'Good manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas.'

Clements, D. (1999)

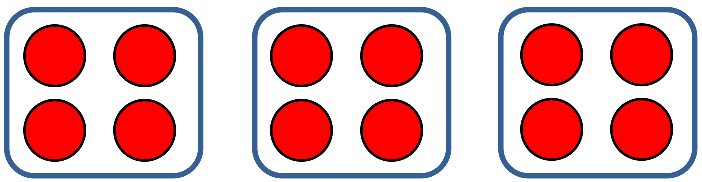
Making connections

$$12 \div 3$$

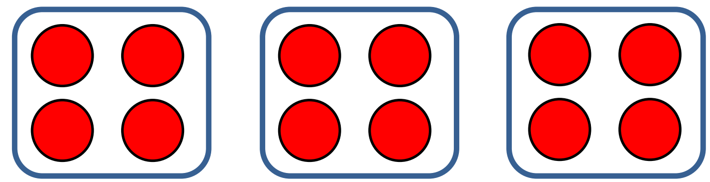


What other questions could we ask?

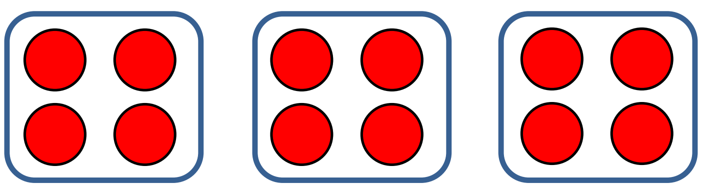
Making connections



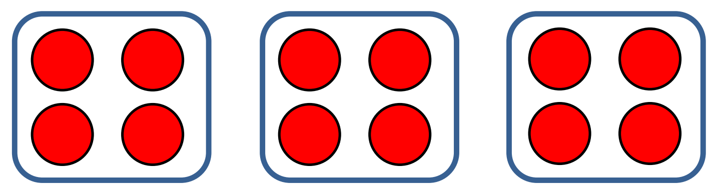
$3 \times 4 = 12$



$4 + 4 + 4 = 12$



$\frac{1}{3}$ of $12 = 4$



Jen and Lucy share 12 sweets in the ratio $1:2$

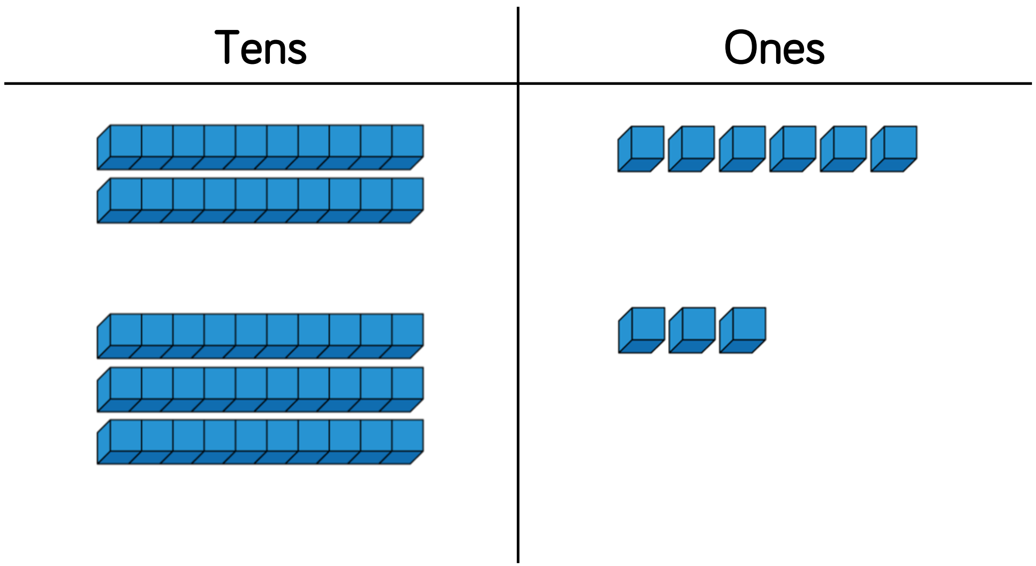
How many sweets does Jen have?

‘A key thing is whether our focus is on finding answers to calculations or ***becoming mindful of the underlying mathematics****.’*

Askew, M. (2012)

Solve... $26 + 33 =$

Model



Calculations

$$\begin{array}{r} 2\ 6 \\ +\ 3\ 3 \\ \hline 5\ 9 \\ \hline \end{array}$$

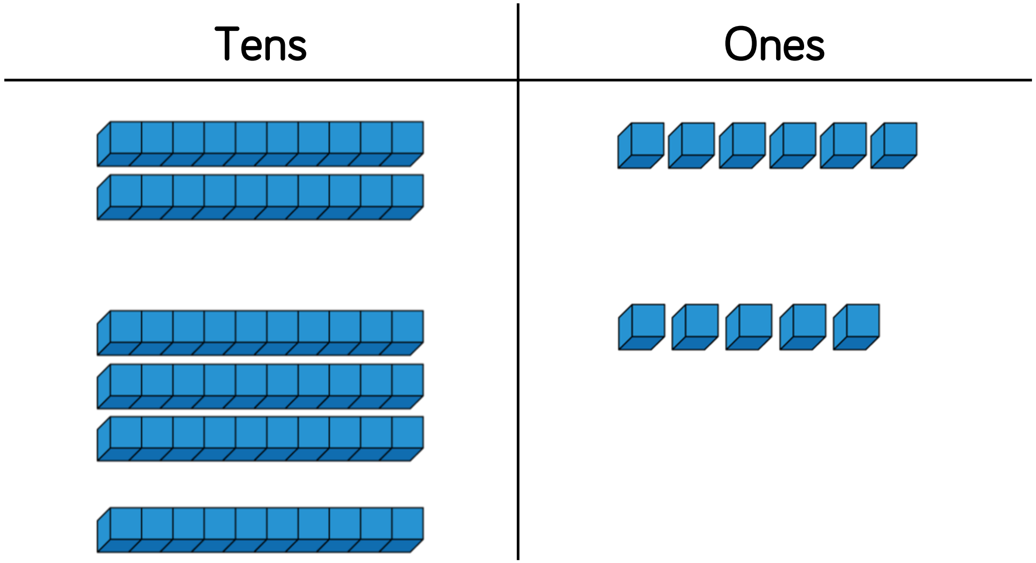
$26 + 35 =$

Key vocabulary: **exchange**

Can we exchange any ones?

Solve... $26 + 35 =$

Model



Calculations

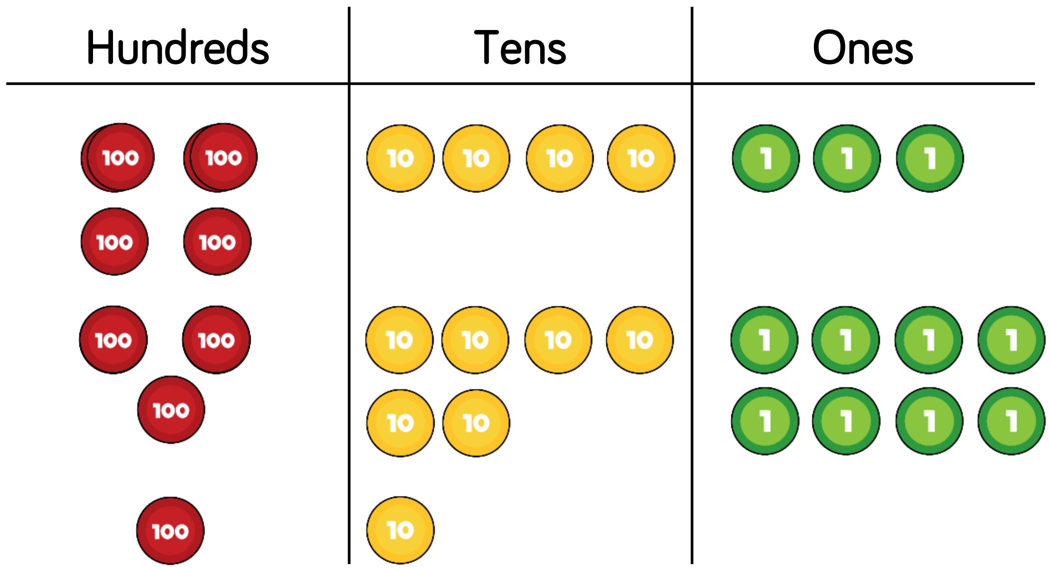
$$\begin{array}{r} 2\ 6 \\ +\ 3\ 5 \\ \hline 6\ 1 \\ \hline {\scriptstyle 1}\ \ \end{array}$$

Key vocabulary: **exchange**

Can we exchange any ones?

Solve... $243 + 368 =$

Model



Calculations

$$\begin{array}{r} 2\;4\;3 \\ +\;3\;6\;8 \\ \hline 6\;1\;1 \\ \hline {\scriptstyle 1\;\;1\;\;} \end{array}$$

$405 + 347 =$

$525 + 378 =$

Key vocabulary: **exchange**

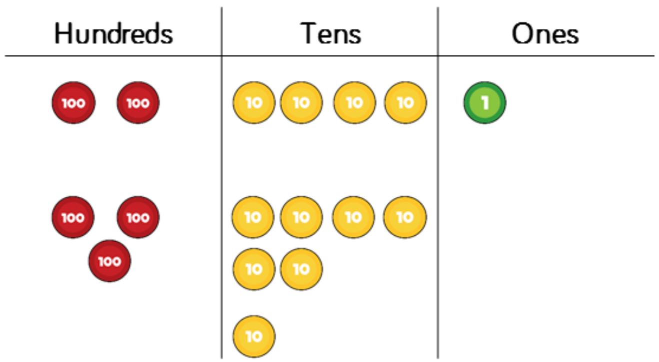
Can we exchange any counters?

Making connections

If there is nothing in a place value column, we use 0 as a place holder.

The written method matches the concrete representation.

Model



Calculations

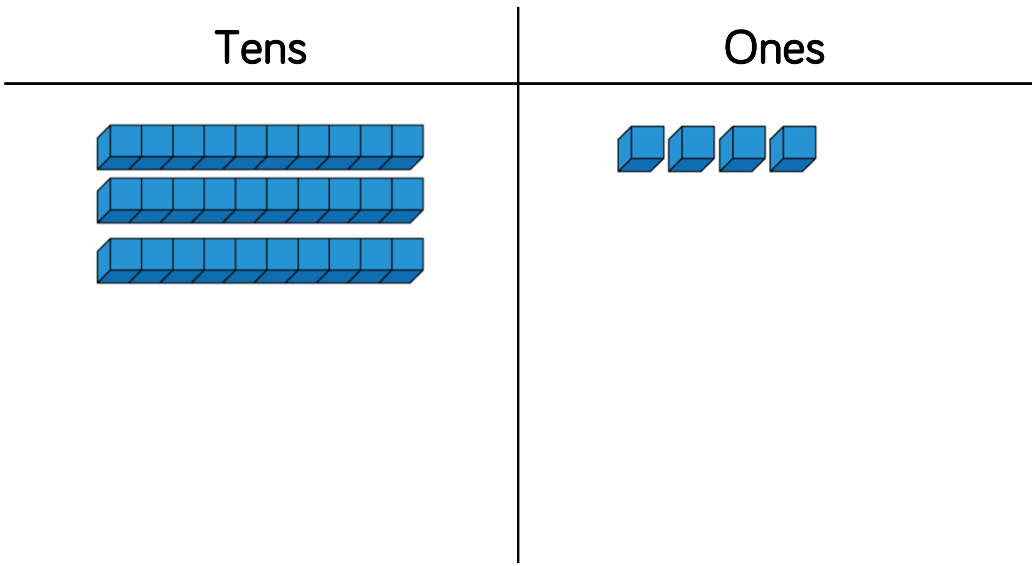
$$\begin{array}{r} 243 \\ +368 \\ \hline 611 \\ \hline {\scriptstyle 1\ 1\ \ } \end{array}$$

Shows exchange when there are 10 in a column.

Start from the right hand column and work to the left.

Solve... $34 - 12 =$

Model



Calculations

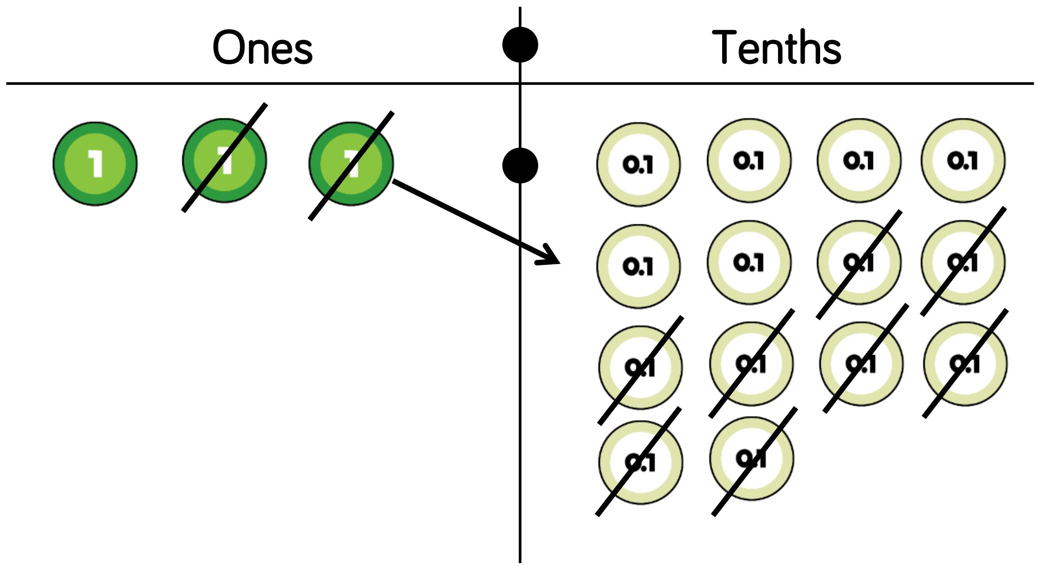
$$\begin{array}{r} 3\ 4 \\ -\ 1\ 2 \\ \hline 2\ 2 \\ \hline \end{array}$$

Key vocabulary: **exchange**

Do we need to exchange any tens?

Solve... $3.4 - 1.8 =$

Model



Calculations

$$\begin{array}{r} {}^{2}\not{3}\,.\,{}^{1}4 \\ -\;1\,.\,8 \\ \hline 1\,.\,6 \\ \hline \end{array}$$

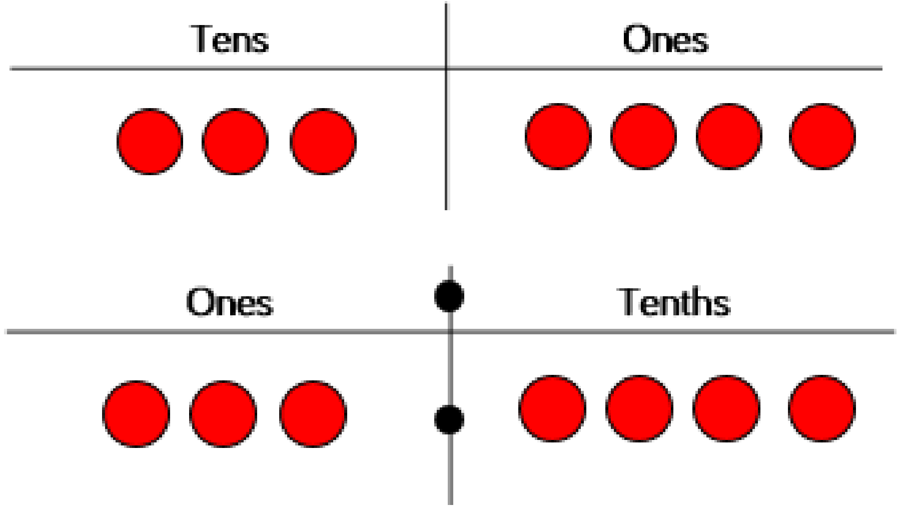
Key vocabulary: **exchange**

Do we need to exchange any ones?

Making connections

Solve... $34 - 18 =$ $3.4 - 1.8 =$

Model



Calculations

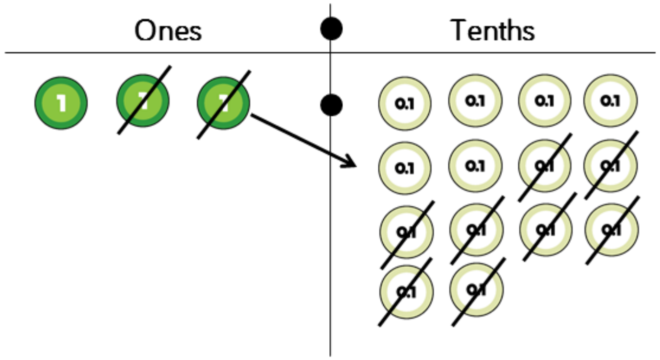
$$\begin{array}{r} 34 \\ -18 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 3.4 \\ -1.8 \\ \hline . \\ \hline \end{array}$$

Making connections

If there is nothing in a place value column, we use 0 as a place holder.

The written method matches the concrete representation.

Model



Calculations

$$\begin{array}{r} {}^{2}\not{3}\,.\,{}^{1}4 \\ -\,1\,.\,8 \\ \hline 1\,.\,6 \end{array}$$

Shows where we can make an exchange.

Start from the right hand column and work to the left.

Solve… $23 \times 6 =$

Model

Calculations

$$
\begin{array}{r}
2\ 3 \\
\times \quad 6 \\
\hline
1\ 3\ 8 \\
\hline
{\scriptstyle 1\ 1\ \ }
\end{array}
$$

$35 \times 5 =$

‘Used well, manipulatives can enable pupils to inquire themselves- becoming independent learners and thinkers. They can also provide a common language with which to communicate cognitive models for abstract ideas.’

Drury, H. (2015)

Understanding the problem

*‘For when the learner has a well-developed symbolic system, it may be possible to by-pass the first two stages. But one does so with the risk that the learner may **not possess the imagery** to fall back on when his **symbolic transformations fail** to achieve a goal in problem solving.’*

Bruner, J. (1966)

Understanding the problem

There are 42 crayons in a box.

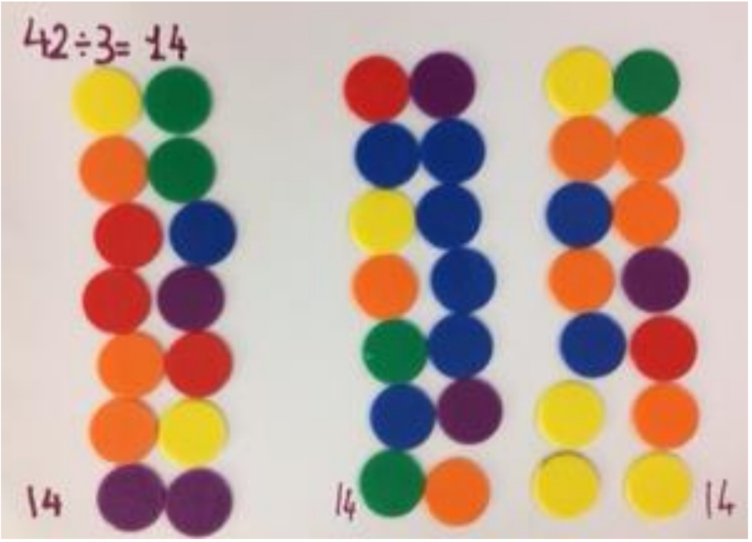
They are shared between three children.

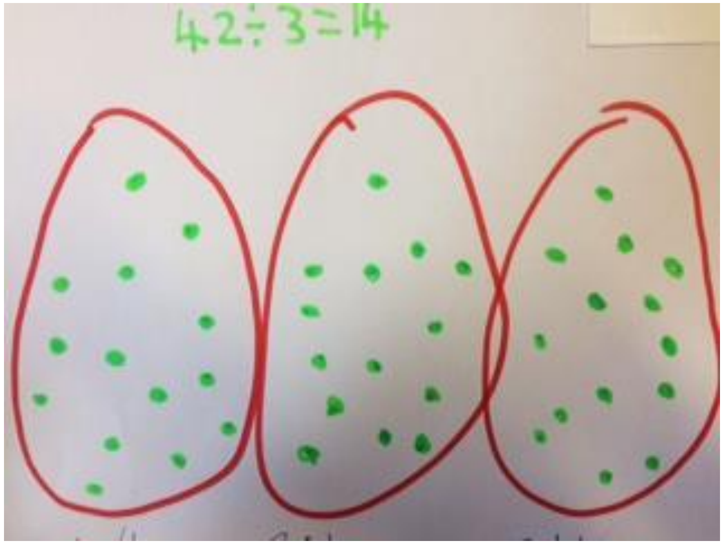
How many crayons do they get each?

How could you represent this problem?

Division - Sharing

There are 42 crayons in a box. They are shared between three children. How many crayons do they get each?





In this version, the children have split 42 into 3 equal groups.

Solve... $42 \div 3 =$

Model

Tens	Ones
10	1 1 1 1
10	1 1 1 1
10	1 1 1 1

Calculations

$$42 \div 3 = 14$$

Understanding the problem

There are 42 crayons in a box.

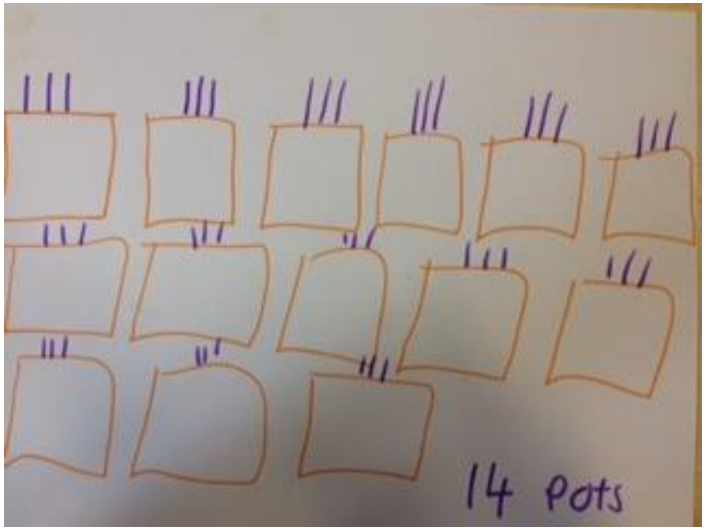
They are put in pots with three in each pot.

How many pots are needed?

How could you represent this problem?

Division - Grouping

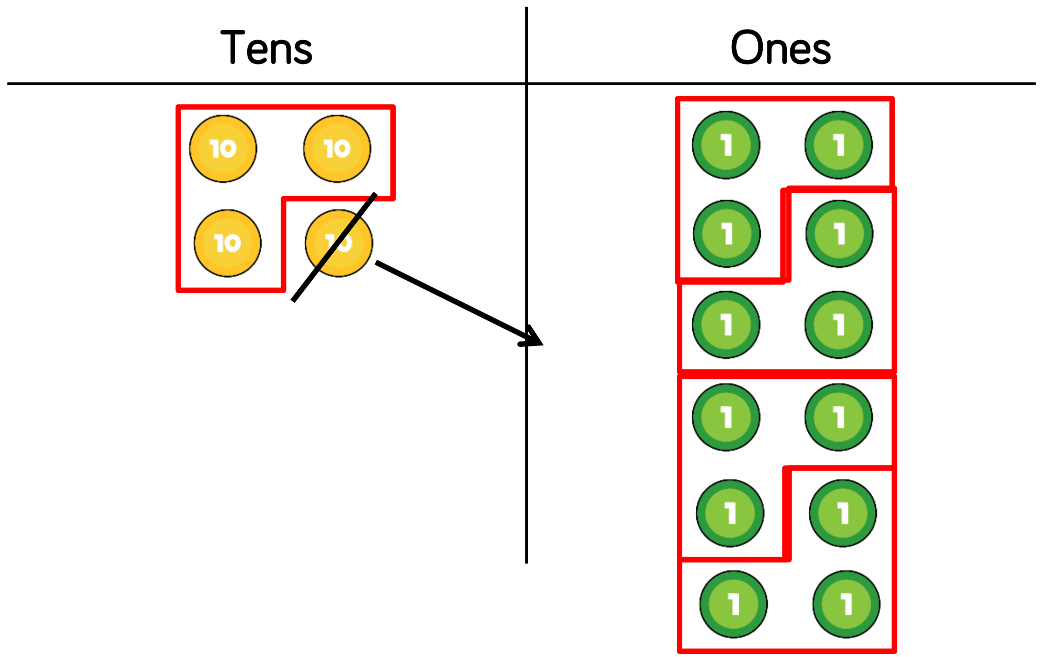
There are 42 crayons in a box. They are put in pots with three in each pot. How many pots are needed?



In this version, the children have counted how many 3s go into 42

Solve… $42 \div 3 =$

Model



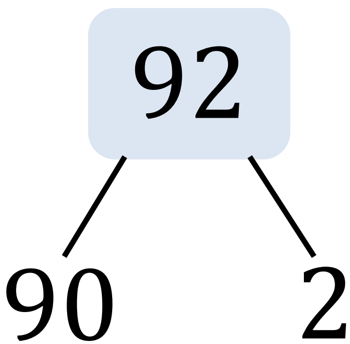
Calculations

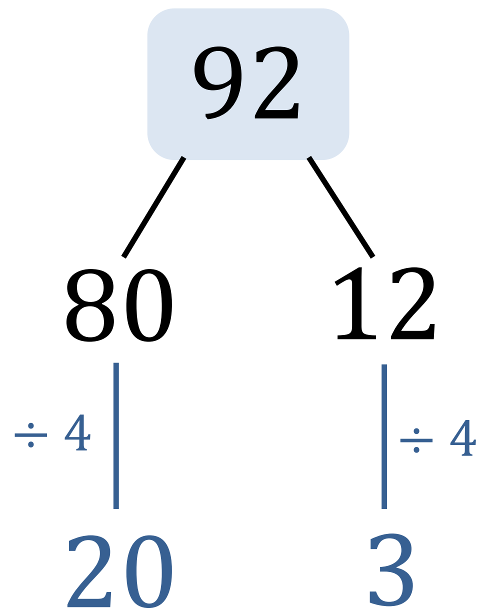
$$\begin{array}{r|l} & 1\ \ 4 \\ \hline 3 & 4\ {}^{1}2 \end{array}$$

$92 \div 4 =$

Division

$92 \div 4 =$





$96 \div 2 =$

$96 \div 3 =$

$96 \div 4 =$

$96 \div 6 =$

$96 \div 8 =$

'These images and models may be viewed as 'stepping stones' within the construction of joint mathematical expertise in relation to a particular task or procedure. They come into their own as the teacher and child negotiate the transition from collaborative activity to internalised individual practice.'

Merttens, R. (2015)

Proving why

Proving why

Make a 2 digit number.

Reverse the digits to make another 2 digit number.

Add the two numbers together.

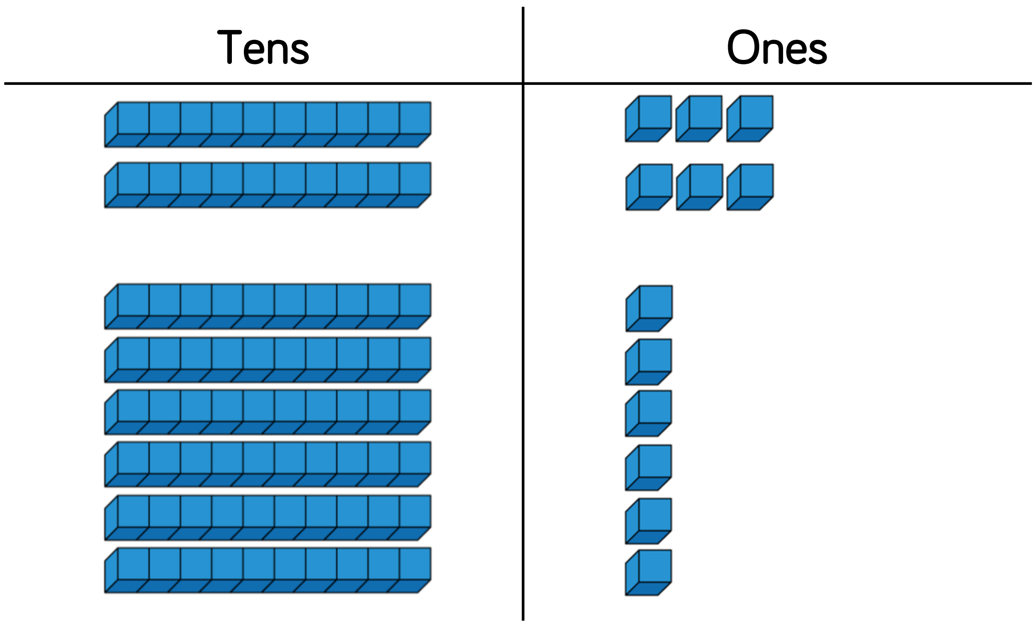
What is the total of the numbers?

Repeat with other numbers.

What do you notice? Why does this happen?

Proving why... $26 + 62 =$

Model



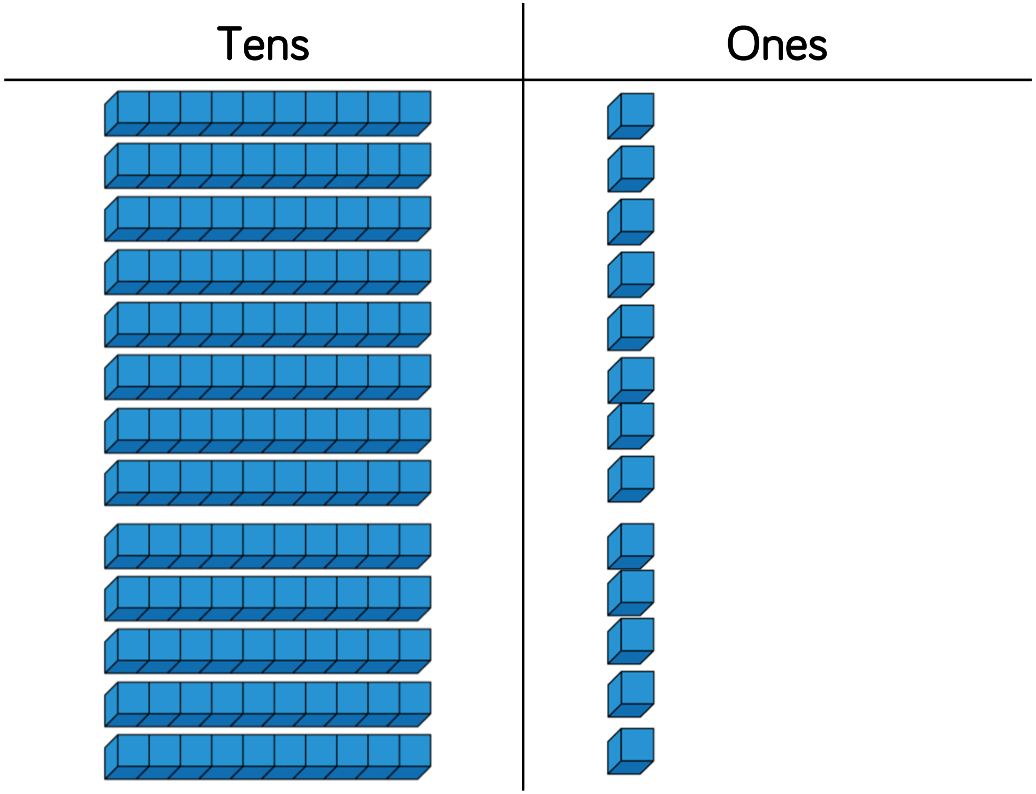
Calculations

$$\begin{array}{r} 26 \\ +\ 62 \\ \hline 88 \\ \hline \end{array}$$

Does this work when we have to exchange in a column?

Proving why... $85 + 58 =$

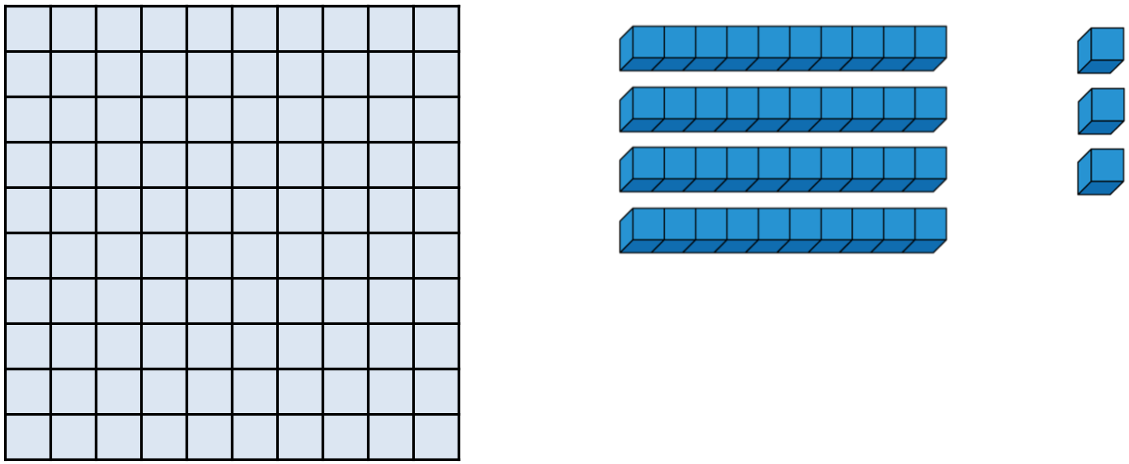
Model



Calculations

$$\begin{array}{r} 8\ 5 \\ +\ 5\ 8 \\ \hline 1\ 4\ 3 \\ \hline {\scriptstyle 1\ \ 1\ \ } \end{array}$$

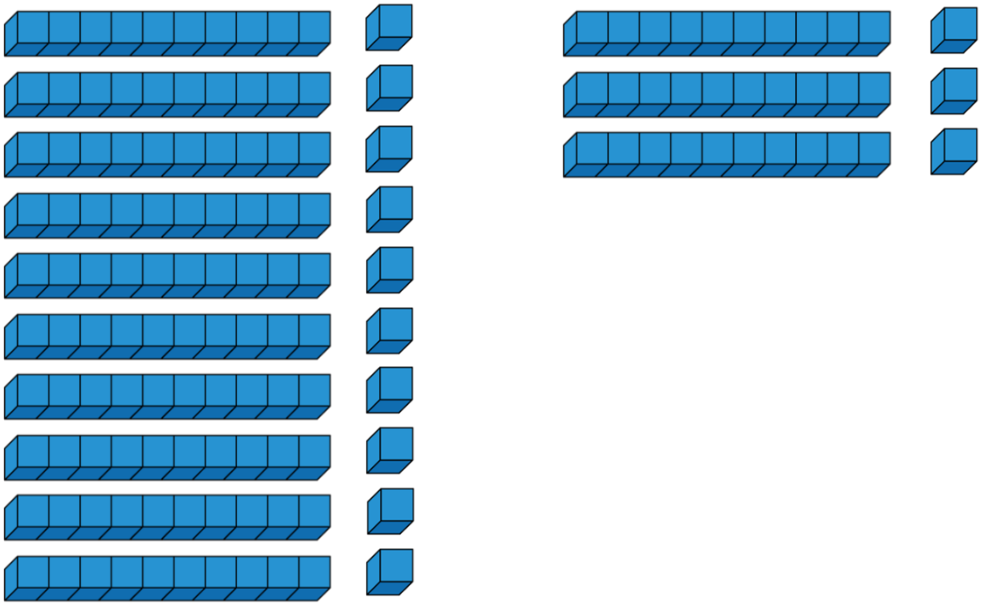
Proving why



$143 = 100 + 40 + 3$

$143 = 1$ hundreds $+ 4$ tens $+ 3$ ones

Proving why



$143 = 14$ tens $+ 3$ ones

$143 = 13$ tens $+13$ ones

Proving why

Make a 2 digit number.

Reverse the digits to make another 2 digit number.

Find the difference between the numbers.

Repeat. What do you notice? Why does this happen?

Prove it using concrete or pictorial representations.

'If we do not use concrete manipulations, then we can not understand mathematics. If we only use concrete manipulations, then we are not doing mathematics.'

Gu, D. (2015)

Any Questions?

Thank you

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