On p-adic zeta functions and their derivatives at $s = 0$

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Abstract

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We study the p-adic interpolation of the special values (suitably regularized) of the Shintani cone zeta functions - the building blocks of standard zeta and L-functions - associated to a real quadratic number field F. Our main result is a polynomial time algorithm to calculate the derivative of these functions of the p-adic variable s at $s = 0$ to high p-adic accuracy. These derivatives are of great interest in view of the classical conjectures of Gross and Stark which express these derivatives at $s = 0$ in terms of certain units in abelian extensions of F.

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Contents

1 Introduction

1.1 Riemann Zeta Function

The Riemann Zeta function $\zeta(s)$ of the complex variable s is defined in the half plane $Re(s) > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1.$$

This function had been studied by Euler, but Riemann was the first to prove that $\zeta(s)$ admits an analytic continuation for all complex s except for a simple pole at $s = 1$. For the special values at non-positive integers $s = 1 - k$, $k = 1, 2, 3, \ldots$, there is the classical formula of Euler,

$$\zeta(1-k) = (-1)^{k+1} \frac{B_k}{k}, \tag{1.1}$$

in terms of the Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, etc, defined by the generating power series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \quad |t| < 2\pi.$$

The rational numbers B_k have many important properties. To give an example, due to Kummer, we fix a prime number p and consider the "p-regularized" zeta function

$$\zeta^*(s) = (1 - p^{-s})\zeta(s) = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1.$$

According to Kummer, the rational numbers

$$\zeta^*(1-k) = (p^{k-1} - 1)\frac{B_k}{k}, \quad k = 0, 1, 2, \ldots,$$

are p-adic integers if $(p-1) \nmid k$. The so-called "Kummer congruences" for Bernoulli numbers are equivalent to

$$\zeta^*(1-k) \equiv \zeta^*(1-k')(mod\, p^{N+1})$$

for any pair of positive integers k, k', such that $k \equiv k'(mod\,(p-1)p^N)$, $(p-1) \nmid k$. In other words, if $1-k$ and $1-k'$ are close p-adically, then so are $\zeta^*(1-k)$ and $\zeta^*(1-k')$. This p-adic interpretation of the Kummer congruences naturally leads to the question, (which in general we will call the "interpolation problem"), as to whether there is a function $\zeta_p(s)$, called the p-adic zeta function, of a p-adic variable s, which is regular outside $s \neq 1$ and which interpolates the special values $\zeta^*(1-k)$, that is,

$$\zeta_p(1-k) = \zeta^*(1-k)$$

if $k > 0$, $(p-1) \nmid k$. That question was raised and answered affirmatively by Kubota-Leopoldt, [42], who established an explicit formula for $\zeta_p(s)$ as a p-adic power series in $s-1$. If the interpolation problem is restricted to those values of $s = 1-k$ which are divisible by $p-1$, then Stark [65] has shown that $\zeta_p(s)$ admits a representation as a p-adic Dirichlet series

$$\zeta_p(s) = \lim_{l \to \infty} \sum_{\substack{n=1 \\ p \nmid n}}^{p^l - 1} n^{-s}, \quad s \in \mathbb{Z}_p.$$

Here, if s is not a positive integer, the power n^{-s} is defined as the p-adic limit of the sequence (n^t) where t runs through a sequence of positive integers in $(p-1)\mathbb{Z}$ in such a way that $t \to -s$ p-adically and $t \to \infty$.

1.2 Hurwitz Zeta Function

Besides the Riemann zeta function, one can also consider the Hurwitz zeta function

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s},\ \ Re(s)>1,$$

where x is a positive rational number. As for the Riemann zeta function $\zeta(s)=\zeta(s,1)$, the Hurwitz zeta function admits an analytic continuation to the whole complex plane except for a simple pole at $s=1$. At $s=1-k$, we have,

$$\zeta(1-k,x)=-\frac{B_k(x)}{k},\ \ k=1,2,3,\ldots, \tag{1.2}$$

where the Bernoulli polynomials $B_k(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}.$$

There is a similar formula for the special values $\zeta^*(1-k,x)$ of the regularized Hurwitz zeta function

$$\zeta^*(s,x)=\sum_{\substack{n=0\\ p\nmid(n+x)}}^{\infty}(n+x)^{-s},\ \ Re(s)>1.$$

Here we assume that x is p-integral and the condition $p\nmid(n+x)$ means $n+x\in\mathbb{Z}_p^*$. The p-adic zeta function $\zeta_p(s,x)$ interpolating the special values $\zeta^*(1-k,x)$, for $k\equiv 0(mod\, p-1)$, $k=1,2,3,\ldots$, is given explicitly in [65] by

$$\zeta_p(s,x)=\lim_{N\underset{p}{\rightarrow}-x}\sum_{\substack{n=0\\ p\nmid(n+x)}}^{N-1}(n+x)^{-s},\ \ s\in\mathbb{Z}_p,$$

where the sequence of positive integers N tends to infinity in such a way that N approaches $-x$ p-adically.

1.3 Zeta functions of number fields

More generally, let F be a number field and E/F be a finite Galois extension of F with abelian Galois group G. By Artin's reciprocity law, each $\sigma \in G$ corresponds to an ideal class C modulo a conductor $f = f_{E/F}$ which is a product of finite and infinite primes of F completely determined by E/F. For any finite set S of places of F which contain at least all places of F ramified in E, the partial Dedekind zeta function attached to this data is defined as

$$\zeta_S(\sigma, s) = \zeta_S(C, s) = \sum_{\substack{\mathfrak{a} \in C \\ (\mathfrak{a}, S)=1}} N(\mathfrak{a})^{-s}, \quad Re(s) > 1.$$

The sum runs over all integral ideals $\mathfrak{a}$ in C which are relatively prime to S. According to Hecke, this zeta function continues analytically to all complex s except $s = 1$. In particular, the special values $\zeta_S(C, 1-k)$, $k = 1, 2, 3, \ldots$, are well-defined. Due to Klingen [39] and Siegel [59] we have the result that the numbers $\zeta_S(C, 1-k)$ are all rational. Moreover, they vanish identically unless F is a totally real number field. For such F, Shintani [56] established an explicit formula for $\zeta_S(C, 1-k)$ in terms of generalized Bernoulli polynomials. The existence of p-adic zeta functions, $\zeta_{S,p}(C, s)$ that solve the interpolation problem for $\zeta_S(C, 1-k)$ was established by Deligne and Ribet [18] and Cassou-Noguès [11] (assuming S contains all primes lying above p). It is worth emphasizing that the p-adic zeta function $\zeta_{S,p}(C, s)$ vanishes unless the number field F is totally real.

The study of the special values of the partial zeta function $\zeta_S(C, s)$ has gained further interest since the arrival of the Stark conjectures ([61], [62], [63], [64])) during the 1970's. In the simplest and most interesting case, $\zeta_S(C, s)$ vanishes at $s = 0$ to the first order. In that case, the conjecture predicts the existence of an algebraic integer

$\epsilon \in E$ such that the derivative at $s = 0$ is given by

$$\zeta_S'(\sigma, 0) = -\frac{1}{w}\, log\, |\sigma(\epsilon)|$$

where w denotes the number of roots of unity in E and the absolute value inside the logarithm is a fixed archimedean place of E lying above an infinite place of F that splits totally in E/F. That conjecture can be viewed as a partial solution to Hilbert's 12-th problem, since it provides a formula for an algebraic number in an abelian extension of the number field F.

In two important papers, [27], [28], Gross proposed an analogue of Stark's conjecture for the p-adic zeta function $\zeta_{S,p}(C, s)$ in the form

$$\zeta_{S,p}'(\sigma, 0) = -log_p\, Norm_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u^\sigma). \tag{1.3}$$

Here S contains all the infinite primes of F, all the places ramified in E and all places of F lying above p. The fundamental assumption is that there exists a prime $\mathfrak{p}$ in F dividing p which splits completely in E. Under this condition, E embeds into $F_{\mathfrak{p}}$ and the conjecture predicts the existence of a $\mathfrak{p}$-unit u in E such that (1.3) holds for all σ in the Galois group $G = Gal(E/F)$. Precise statements of the conjectures of Stark and Gross are given in the Appendix.

The conjecture of Gross raises the question how to calculate the numbers $\zeta_{S,p}'(\sigma, 0)$ to a high level of p-adic accuracy. Our thesis is devoted to the study of that problem.

1.4 Approach and Results

The perspective we take on the Gross conjecture is to find a formula for calculating the left side of Equation (1.3) to many p-adic digits. One could then, for particular field extensions E/F as described, consider a (finite) number of likely candidates for the element u on the right side of the equation and potentially verify the conjecture is true in this particular case. By the work of Shintani, reviewed below, partial Dedekind zeta functions can be reformulated entirely in terms of numbers rather than ideals. Accordingly, the problem is to construct parallel p-adic zeta functions which interpolate the values of regularized zeta functions constructed from complex zeta functions of the form (due to Shintani)

$$\zeta(s,x;A) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{j=1}^{n} (\sum_{i=1}^{r} (m_i + x_i) a_{i,j})^{-s}, \;\; Re(s) > \frac{r}{n} \tag{1.4}$$

where $A = (a_{i,j})$ and $x = (x_1, \ldots, x_r)$. These zeta functions have r summation variables and n linear forms. We could, for example, refer to the case of F a quadratic field as the $r = 2$, $n = 2$ case. By the work of Cassou-Noguès [11] we know the related p-adic zeta functions exist. The problem is to construct them in such a way that we can calculate their derivatives at $s = 0$ to as many p-adic digits as we wish. Constructing them is, of course, the same as solving the particular interpolation problem.

In [65], Stark took a direct approach to the definition of $\zeta_p(s, x; A)$ in the case $r = 1$, $n = 1$. He showed it solved the interpolation problem and he showed how to calculate its derivative at $s = 0$ in terms of a p-adic gamma function. Here the base field is $F = \mathbb{Q}$ and the question whether p splits or is inert in F does not arise. In [66], Stark completed the case of one linear form and one summation variable by giving an elementary formula for the p-adic expansion of $\zeta_p'(0, x; A)$ to as many p-adic digits as we wish. His work, and particularly the talk he gave at the Baltimore conference [66], are the motivation for this thesis.

In this thesis we extend the work of Stark to the definition of p-adic zeta functions for the case of two summation variables and both one and two linear forms (the $r = 2$, $n = 1$ and $r = 2$, $n = 2$ cases). We show they solve the interpolation problem, and we show how to calculate their derivatives at $s = 0$ to many p-adic digits (the main theorems). These results cover the case of F a quadratic number field where the chosen prime p splits in F.

The organization of this thesis is as follows. In Section (2) we review the work of Shintani in expressing the partial Dedekind zeta functions in the form of numbers rather than ideals. We also review the direct approach of Stark to the $r = 1$, $n = 1$ case. In Section (3) we define the general zeta functions of the form (1.4) and extend the known results given by equations (1.1) and (1.2) to all $r > 1$, including defining generalized Bernoulli polynomials for use in calculating the special values and showing how, in turn, to calculate these generalized Bernoulli polynomials for given values of their variables. In Section (4) we use the direct approach to construct p-adic zeta functions in the two cases, $r = 2$, $n = 1$ and $r = 2$, $n = 2$. We then show how to find their derivatives at $s = 0$ to many p-adic digits. All our results are for the case where the chosen prime p splits in the given number field F.

Throughout this thesis, our p-adic functions will be for odd primes p only. As is usually the case in p-adic analysis, the case $p = 2$ requires (sometimes lengthy) modifications.

2 Basic concepts and results

2.1 Results of Shintani

The result due to Shintani [56] in expressing the partial Dedekind zeta functions in terms of numbers rather than ideals is as follows. Let F be a totally real algebraic number field of degree n over $\mathbb{Q}$. Let $\mathfrak{f}$ be an integral ideal of F, that is, an ideal in the ring of integers $\mathbb{Z}_{\mathbb{F}}$ of F. Two integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F are called equivalent modulo $\mathfrak{f}$, that is, $\mathfrak{a} \equiv \mathfrak{b}(mod\ \mathfrak{f})$, if and only if,

(1) $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime to $\mathfrak{f}$,

(2) $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal, that is, $\mathfrak{a}\mathfrak{b}^{-1} = (\alpha)$, $\alpha \in F$,

(3) the generator α is chosen such that $\alpha \equiv 1(\mathfrak{f}\mathfrak{b}^{-1}) \Leftrightarrow \alpha - 1 \in \mathfrak{f}\mathfrak{b}^{-1}$,

(4) $\alpha \gg 0$ or α is totally positive, that is, for every embedding σ_j of F into $\mathbb{R}$, $\sigma_j(\alpha) > 0$.

The equivalence classes are ideal classes forming the narrow ray class group modulo $\mathfrak{f}$, denoted $Cl_F(\mathfrak{f})$.

The Dedekind zeta function is defined by the absolutely convergent Dirichlet series,

$$\zeta_F(s) = \sum_{\substack{\mathfrak{a} \subseteq \mathbb{Z}_F \\ \mathfrak{a} \neq 0}} N(\mathfrak{a})^{-s}, \qquad Re(s) > 1. \tag{2.1}$$

Let E be an abelian extension of F with $G = Gal(E/F)$ and let $\mathfrak{f}$ be the conductor of E/F. Let $\sigma \in G$. The restriction of the summation in (2.1) to those ideals $\mathfrak{a}$ prime to $\mathfrak{f}$ such that $\sigma_{\mathfrak{a}} = \sigma$ where $\sigma_{\mathfrak{a}}$ is the element of G associated with $\mathfrak{a}$ by Artin reciprocity, defines the partial zeta function,

$$\zeta_F(\sigma, s) = \sum_{\substack{\mathfrak{a}\subseteq\, \mathbb{Z}_F,\ (\mathfrak{a},\mathfrak{f})=1 \\ \sigma_{\mathfrak{a}}=\sigma,\ \mathfrak{a}\neq 0}} N(\mathfrak{a})^{-s}. \tag{2.2}$$

Then, with $C = \{\mathfrak{a} \subseteq \mathbb{Z}_F \mid (\mathfrak{a}, \mathfrak{f}) = 1,\ \sigma_{\mathfrak{a}} = \sigma\}$ we have, by the Artin Reciprocity Law, the equivalent definition

$$\zeta(C, s) = \zeta_{E/F}(C, s) = \sum_{\mathfrak{a}\in C} N(\mathfrak{a})^{-s} = \sum_{\substack{\mathfrak{a}\subseteq\, \mathbb{Z}_F \\ \mathfrak{a}\in C,\ \mathfrak{a}\neq 0}} N(\mathfrak{a})^{-s}, \qquad Re(s) > 1. \tag{2.3}$$

Now, let us fix a particular congruence class C and choose a representative integral ideal $\mathfrak{b}$ of C. Then, for these mutually prime ideals $\mathfrak{b}$ and $\mathfrak{f}$ of F, we can set

$$\zeta(C, s) = \zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\mathfrak{a}\in C} N\mathfrak{a}^{-s},$$

where the summation is over all integral ideals $\mathfrak{a}$ of F which are in the same narrow ray class group modulo $\mathfrak{f}$ (denoted by C) as $\mathfrak{b}$. Then we can write

$$\zeta(C, s) = \sum_{\mathfrak{a}\in C} N\mathfrak{a}^{-s} = N\mathfrak{b}^{-s} \sum_{\substack{(\alpha) \\ \alpha\in 1+\mathfrak{f}\mathfrak{b}^{-1} \\ \alpha>>0}} N((\alpha))^{-s} = N\mathfrak{b}^{-s} \sum_{\substack{(\alpha) \\ \alpha\in 1+\mathfrak{f}\mathfrak{b}^{-1} \\ \alpha>>0}} N(\alpha)^{-s}$$

since α is totally positive.

Now, $(\alpha) = (\beta) \Leftrightarrow \alpha = \beta\varepsilon$ where ε is a unit in F, so summation over ideals can be replaced by summation over numbers modulo the units as follows. Let $U(\mathfrak{f})_+$ be the group of totally positive units of F that are congruent to 1 *mod* $\mathfrak{f}$. Then $U(\mathfrak{f})_+$ is a subgroup of finite index of the group U_+ of totally positive units of F. Then

$$\zeta(C,s) = N\mathfrak{b}^{-s} \sum_{\substack{\alpha \in 1+\mathfrak{f}\mathfrak{b}^{-1} \\ \alpha \; mod \; U(\mathfrak{f})_+ \\ \alpha >> 0}} N(\alpha)^{-s} \tag{2.4}$$

Let $\mathfrak{D}$ be a fundamental domain for the action of $U(\mathfrak{f})_+$ on $\mathbb{R}^n_{>0}$. Following Shintani [56] we choose $\mathfrak{D}$ to be a disjoint union of open simplicial cones, $\mathfrak{D} = \bigsqcup_{k \in T} C_k$, where T is a finite set of indices and $C_k = C_k(a_{1,k}, \ldots, a_{r(k),k})$, $k \in T$, with generators $a_{1,k}, \ldots, a_{r(k),k}$ which we can choose to be in $\mathfrak{f}\mathfrak{b}^{-1} \subset F$. Then,

$$\zeta(C,s) = N\mathfrak{b}^{-s} \sum_{k \in T} \sum_{\alpha \in (1+\mathfrak{f}\mathfrak{b}^{-1}) \cap C_j} N(\alpha)^{-s}$$

But the $a_{j,k}$ are algebraic numbers in F. Hence they form a $\mathbb{Q}$ -basis of F. So every $\alpha \in F$ has a unique representation in terms of the $a_{j,k}$, namely,

$$\alpha = \sum_{i=1}^{r(k)} \frac{a_i}{b_i} a_{i,k}, \quad a_i,\ b_i \in \mathbb{Z}, \quad b_i \neq 0.$$

So we can write,

$$\alpha = \sum_{i=1}^{r(k)} (m_i + x_i) a_{i,k}$$

with $m_1, \ldots, m_{r(k)} \in \mathbb{Z}_+$ and $x_1, \ldots, x_{r(k)} \in \mathbb{Q}$, $x_1, \ldots, x_{r(k)} \in [0,1)$. Then,

$$\zeta(C,s) = N\mathfrak{b}^{-s} \sum_{k \in T} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{r(k)}=0}^{\infty} N(\sum_{i=1}^{r(k)} (m_i + x_i) a_{i,k})^{-s}. \tag{2.5}$$

To begin, if for example, we take $k = 1$ in the first sum of Equation (2.5) and put

$r = r(1)$, we can simply consider

$$\sum_{m1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} N(\sum_{i=1}^{r}(m_i + x_i)a_{i,1})^{-s} = \sum_{m1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{j=1}^{n} (\sum_{i=1}^{r}(m_i + x_i)a_{i,j}),$$

where $a_{i,j}$ is the j-th embedding of $a_{i,1}$ in $\mathbb{C}$.

We note the columns of the $r \times n$ matrix $A = \begin{pmatrix} a_{1,1} & \dots & a_{1,j} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{r,1} & \dots & a_{r,j} & \dots & a_{r,n} \end{pmatrix}$

are the n embeddings of $a_{1,1}, \dots, a_{r,1}$ in $\mathbb{C}$. So we define,

$$\zeta(s, x; A) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{j=1}^{n} \big(\sum_{i=1}^{r}(m_i + x_i)a_{i,j}\big)^{-s}, \quad Re(s) > \frac{r}{n}. \tag{2.6}$$

We can then write

$$\zeta(C, s) = N\mathfrak{b}^{-s} \sum_{k \in T} \zeta(s, x(k); A(k)) \tag{2.7}$$

for $x = x(k)$ and $A = A(k)$. Shintani [56] and [57] has given explicit formulas for $\zeta(s, x; A)$ at $s = 1 - m$, $m = 1, 2, \dots$ and, in the $r = 2$, $n = 2$ case, for $\zeta'(0, x, A)$.

We may therefore refer to zeta functions of the type given by (2.6) as Shintani cone zeta functions.

2.2 Stark's direct approach to p-adic interpolation

The usual (indirect) approach to p-adic interpolation of zeta functions $\zeta(s)$ defined for $s \in \mathbb{C}$, with $Re(s) > r$ for some positive integer r, is to delete the terms divisible by p to give the regularized function $\zeta^*(s)$, show analytic continuation to all values $s \in \mathbb{C}$ apart from the finite number of poles, find the (special) values at non-positive integers k, determine the p-adic function $\zeta_p(-k)$ which agrees with all of these values, or at least on a set of non-negative integers which is a dense subset of $\mathbb{Z}_p$, and conclude $\zeta_p(s)$ is the unique p-adic function that interpolates these special values of $\zeta(s)$. The existence of this unique p-adic function is due to the work of Kubota and Leopoldt [42], also Iwasawa [36]. Stark's (direct) approach, in the $r = 1$, $n = 1$ case, was to simply define a p-adic Dirichlet series and then show it satisfies the required conditions.

In [65] and [66], Stark argues as follows. The standard Riemann zeta-function and the related L-function are given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

for $s \in \mathbb{C}$, $Re(s) > 1$, χ a Dirichlet character mod m, $m \geq 2$. Both have analytic continuation to all of $s \in \mathbb{C}$ apart from a simple pole at $s = 1$.

In [65], Stark defines for $S = \{\infty, p\}$, p a prime, $s \in \mathbb{C}$, the zeta and related L-functions,

$$\zeta(s,S) = \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} n^{-s}, \quad L(s,\chi,S) = \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \chi(n) n^{-s}.$$

We have,

$$L(s,\chi,S) = L(s,\chi)(1 - \chi(p)^{-1} p^{-s}). \tag{2.8}$$

We also have for f a positive rational integer and χ a Dirichlet character mod f,

$$L(s,\chi,S) = \sum_{a=1}^{f-1} \chi(a) f^{-s} \zeta(s, \frac{a}{f}, S),$$

where we introduce the partial zeta-function,

$$\zeta(s, \frac{a}{f}, S) = f^s \sum_{\substack{n \equiv a (mod\ f) \\ (n,p)=1}} n^{-s} = \sum_{\substack{n=0 \\ (nf+a,p)=1}}^{\infty} (n + \frac{a}{f})^{-s}.$$

By Equation (2.8), the partial zeta function satisfies:

$$\zeta(s, \frac{a}{f}, S) = \zeta(s, \frac{a}{f}) - p^{-s}\zeta(s, \frac{b}{f}), \text{ for } b \text{ such that } bp \equiv a(mod\ f),\ \ 0 < b < f.$$

The Hurwitz zeta function is defined by,

$$\zeta(s, x) = \sum_{n=0}^{\infty} (n + x)^{-s},\ \ x > 0,\ \ Re(s) > 1.$$

By Corollary (11), this function has analytic continuation to all of $s \in \mathbb{C}$ apart from a pole at $s = 1$ with residue 1, and at non-negative integers k, $\zeta(-k, x) = -\frac{B_{k+1}(x)}{k+1}$, which gives,

$$\zeta(-k, \frac{a}{f}, S) = -\frac{B_{k+1}(\frac{a}{f})}{k+1} + p^k \frac{B_{k+1}(\frac{b}{f})}{k+1}, \text{ for } bp \equiv a(mod\ f),\ \ 0 < b < f. \tag{2.9}$$

In [65], Stark took a different approach to the p-adic interpolation of these zeta-functions. He first took the usual definition of n^{-s} for $s \in \mathbb{Z}_p$, namely

$$n^{-s} = \lim_{k \to -s} n^k$$

where k runs through a sequence of integers congruent to $0(mod\, p-1)$ and tending to $-s$ p-adically. Then he simply defined the p-adic zeta-function,

$$\zeta_p(s, x, S) = \lim_{N \xrightarrow[p]{} -x} \sum_{\substack{0 \le n < N \\ n+x \in \mathbb{Z}_p^+}} (n + x)^{-s},$$

and then proved it is defined and continuous for all s and $x \in \mathbb{Z}_p^+$ and that

$$\zeta_p(-k, \frac{a}{f}, S) = -\frac{B_{k+1}(\frac{a}{f}) - B_{k+1}(0)}{k+1} + p^k\left(\frac{B_{k+1}(\frac{b}{f}) - B_{k+1}(0)}{k+1}\right) \tag{2.10}$$

where b is such that $bp \equiv a(mod\ f)$, $0 < b < f$. Since $B_{k+1}(0) = 0$ for even k, the equality of the right hand sides of Equations (2.9) and (2.10) on a dense set $k \in \mathbb{Z}_p$ allows us to conclude that $\zeta_p(s, \frac{a}{f}, S)$ is the unique p-adic function which p-adically interpolates the values of $\zeta(s, \frac{a}{f}, S)$ for s a non-positive integer. We then have,

$$\zeta_p(0, \frac{a}{f}, S) = -B_1(\frac{a}{f}) + B_1(\frac{b}{f}) = \frac{b}{f} - \frac{a}{f}, \text{ for } bp \equiv a(mod\ f),\ \ 0 < b < f. \tag{2.11}$$

Stark further defined a p-adic gamma function,

$$\Gamma_p(x)^{-1} = \lim_{N \underset{p}{\longrightarrow} -x} (-1)^{pN} \prod_{\substack{n<N \\ (n+x)_p=1}} (n+x).$$

He showed his gamma function is equivalent to the Morita p-adic gamma function definition,

$$\Gamma_p(x) = \lim_{M \underset{p}{\longrightarrow} x} (-1)^{pM} \prod_{\substack{n<M \\ (n,p)=1}} (n+x),$$

and easily proved the key relationship,

$$\zeta_p'(0, x, S) = \log_p \Gamma_p(x). \tag{2.12}$$

The same proofs can be used to show the rescaled Hurwitz p-adic zeta and p-adic gamma functions introduced in [66], namely,

$$\zeta_p(s, x, f) = \lim_{N \underset{p}{\longrightarrow} -\frac{x}{f}} \sum_{\substack{0 \le n < N \\ (n,p)=1}} (nf + x)^{-s}, \text{ and,} \tag{2.13}$$

$$\Gamma_p(x,f)^{-1} = \lim_{N \xrightarrow[p]{} -\frac{x}{f}} (-1)^{pN} \prod_{\substack{n<N \\ p \nmid (nf+x)}} (nf+x) \tag{2.14}$$

satisfy the same key relationship,

$$\zeta_p'(0,x,f) = \log_p \Gamma_p(x,f). \tag{2.15}$$

In [66], Stark showed how to calculate $\zeta_p'(0,x,f)$ to many p-adic digits. Accordingly, we used the algorithm in GP-Pari to calculate $log_p \Gamma_p(x,f)$ and Stark's result to calculate $\zeta_p'(0,x,f)$ and we were able to verify agreement to many p-adic digits (e.g., to p^{50} for small values of p) for many different values of p, f and x.

Unfortunately, the simplicity of this $r=1$, $n=1$ case does not extend to zeta functions with more that one summation variable ($r>1$) and therefore not to zeta functions with more than one linear form ($n>1$).

3 Complex zeta functions

We begin with a general definition for the complex zeta functions with which we will be concerned. We prove analytic continuation for these zeta functions to the whole complex plane, except for a finite number of poles, and we obtain their (special) values at negative integers in terms of a combination of generalized Bernoulli polynomials.

3.1 Definition

Let $A = (a_{i,j})$ be an $r \times n$ matrix with the $a_{i,j}$ positive real numbers. We denote by $A^{i,j}_{k,l}$ the following sub-matrix of A,

$$A^{i,j}_{k,l} = \begin{pmatrix} a_{i,j} & \dots & a_{i,l} \\ \dots & \dots & \dots \\ a_{k,j} & \dots & a_{k,l} \end{pmatrix}.$$

In particular, $A = A^{1,1}_{r,n}$. We put $\mathbb{Z}_0 = \{0, 1, \ldots\}$. We denote by L_j, $j = 1, \ldots, n$, the linear form in r variables given by $L_j(t_1, \ldots, t_r) = \sum_{i=1}^{r} a_{i,j} t_i$. Then, we define a generalized complex ζ-function with r summation variables and n linear forms as follows.

Definition 1. *For a vector $x = (x_1, \ldots, x_i, \ldots, x_r)$, $x_i \in \mathbb{R}^+ \cup \{0\}$, $x \neq 0$, let*

$$\zeta(s, x; A) = \sum_{m \in \mathbb{Z}_0^r} \prod_{j=1}^{n} L_j(m + x)^{-s}, \quad Re(s) > \frac{r}{n}. \tag{3.1}$$

For $x = 0$ we define,

$$\zeta(s, 0; A) = \sum_{\substack{m \in \mathbb{Z}_0^r \\ m \neq 0}} \prod_{j=1}^{n} L_j(m)^{-s}, \quad Re(s) > \frac{r}{n}. \tag{3.2}$$

The convergence of this Dirichlet series is well-known.

We will also use the notation $w = xA = (w_1, \dots, w_j, \dots, w_n)$, $w_j = \sum_{i=1}^r a_{i,j}x_i$, to express our zeta function for $x \neq 0$ in either of the three ways:

$$\begin{aligned}\zeta(s, x; A) &= \sum_{m\in\mathbb{Z}_0^r} \prod_{j=1}^n L_j(m+x)^{-s} \\ &= \sum_{m\in\mathbb{Z}_0^r} \prod_{j=1}^n \left(\sum_{i=1}^r (m_i + x_i)a_{i,j}\right)^{-s} \\ &= \sum_{m\in\mathbb{Z}_0^r} \prod_{j=1}^n \left(w_j + \sum_{i=1}^r m_i a_{i,j}\right)^{-s}.\end{aligned}$$

In the latter case, we may also use the notation $\zeta(s, w_j; A_j)$ rather than $\zeta(s, x; A_j)$. This will occur later when we want to use the results below that the special values of the analytically continued zeta functions may be viewed either as polynomials in the components of w or as polynomials in the components of x.

3.2 Complex zeta functions with one linear form

Let us first consider a zeta function with one linear form and r summation parameters. To simplify notation, we let $A_j = (a_{1,j}, \dots, a_{r,j})^t$ be the j-th column of A. Then we have

$$\begin{aligned}\zeta(s, x; A_j) = \zeta(s, x; A_{r,j}^{1,j}) &= \sum_{m_1=0}^{\infty} \dots \sum_{m_r=0}^{\infty} L_j(m+x)^{-s} \\ &= \sum_{m_1=0}^{\infty} \dots \sum_{m_r=0}^{\infty} \left(\sum_{i=1}^r (m_i + x_i)a_{i,j}\right)^{-s} \\ &= \sum_{m_1=0}^{\infty} \dots \sum_{m_r=0}^{\infty} \left(w_j + \sum_{i=1}^r m_i a_{i,j}\right)^{-s},\end{aligned}$$

where $w_j = xA_j = \sum_{i=1}^{r} x_i a_{i,j}$.

3.2.1 Analytic continuation, values at negative integers

We generalize results given by several authors including Ahlfors [1], page 214, and Apostol [2], page 251, which use the Riemann "loop or keyhole integral" (see Riemann [51] translated and published in the appendix of Edwards [20]). After first showing $\zeta(s, x; A_j)$ has a meromorphic continuation to the whole complex plane, our particular interest is in the special values of $\zeta(-k, x; A_j)$, where $k = 0, 1, 2, \ldots$. These special values may be presented in different ways. The following theorem first presents the special values in the general case of r summation variables and one linear form, in a notationally compact manner which is useful in certain investigations (e.g., Stark [66]). The corollary following the theorem presents the special value at $k = 0$ in the case of two summation variables and one linear form (r=2, $n = 1$) in a form which is notationally less compact but more useful in other investigations (e.g., Shintani [56]).

Theorem 2. *The Dirichlet series $\zeta(s, x; A_j)$ converges for $s > r$ and analytically continues to the whole complex plane except for simple poles at $s = 1, \ldots, r$ with residue at $s = r$ given by,*

$$Res_{s=r}\zeta(s, x; A_j) = \frac{1}{(r-1)!} \frac{1}{\prod_{i=1}^{r} a_{i,j}}. \tag{3.3}$$

Moreover, the special values at $s = -k$, k a non-negative integer, are given by

$$\zeta(-k, x; A_j) = (-1)^k \frac{k!}{(k+r)!} B_{k+r}(\bar{w}_j; A_j) = (-1)^r \frac{k!}{(k+r)!} B_{k+r}(w_j; A_j), \tag{3.4}$$

where $\bar{w}_j = \sum_{i=1}^{r}(1 - x_i)a_{i,j}$ and the generalized Bernoulli polynomial $B_j(y; A_j)$ is *defined by the generating function*

$$\frac{t^r e^{yt}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)} = \sum_{j=0}^{\infty} \frac{B_j(y;A_j)}{j!} t^j,$$

Proof. We first prove the analytic continuation statement.

Let $w_j = xA_j = \sum_{l=1}^{r} a_{l,j}x_l$ and $\bar{w}_j = \sum_{l=1}^{r}(1-x_l)a_{l,j}$. The Euler gamma function is defined by,

$$\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}\,dy, \qquad Re(s) > 0.$$

Substituting $y = (w_j + \sum_{l=1}^{r} m_l a_{l,j})\,t$, we have,

$$\Gamma(s) = \left(w_j + \sum_{l=1}^{r} m_l a_{l,j}\right)^s \int_0^{\infty} e^{-\left(w_j+\sum_{l=1}^{r} m_l a_{l,j}\right)t} t^{s-1} dt, \text{ so that,}$$

$$\begin{aligned}
\Gamma(s)\zeta(s,x;A_j) &= \int_0^{\infty} e^{-w_j t} \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} e^{-\sum_{l=1}^{r} m_l a_{l,j} t}\ t^{s-1}\ dt \\
&= \int_0^{\infty} \frac{e^{-w_j t}}{\prod_{l=1}^{r}(1-e^{-a_{l,j}t})}\ t^{s-1}\ dt \\
&= \int_0^{\infty} \frac{e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)}\ t^{s-1}\ dt.
\end{aligned}$$

Since $\frac{e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)} = O(t^{-r})$ at $t = 0$ and $\Gamma(s)$ converges for $Re(s) > 1$, the Dirichlet series converges absolutely for $s > r$.

Consider

$$I(s,x;A_j) = \int_C \frac{z^{s-1}e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\,dz$$

where the path C is the sum of the three paths in $\mathbb{C}$:

C_1: the interval $[\infty, \rho]$

C_2: the counterclockwise circle of radius ρ around the origin

C_3: the interval $[\rho, \infty]$.

We can write

$$I(s,x;A_j) = I_1(s,x;A_j) + I_2(s,x;A_j) + I_3(s,x;A_j)$$

where $I_k(s,x;A_j) = \int_{C_k} \frac{e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\ z^{s-1}\ dz$ for k= 1, 2, 3.

Now, with $s = \sigma + it$, on C_2, $|z^{s-1}| = |z^{\sigma-1+it}| \leq |z|^{\sigma-1} = \rho^{\sigma-1}$, to give,

$$|I_2(s,x;A_j)| = \left| \int_{|z|=\rho} \frac{z^{s-1}e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\, dz \right| \leq \rho^{\sigma-r-1} \sup_{|z|=\rho} \left| \frac{z^r e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)} \right| 2\pi\rho.$$

But, $\dfrac{e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}$ is analytic in $|z| < 2\pi$ except for a pole of order r at $z = 0$,

so $\dfrac{z^r e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{-a_{l,j}z}-1)}$ is analytic, and $\left| \dfrac{z^r e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{-a_{l,j}z}-1)} \right| \leq B$ for some constant B.

Hence, $|I_2(s,x;A_j)| \to 0$ as $\rho \to 0$ for $\sigma > r$. Now, (using t for the real axis variable) since $z^{s-1} = t^{s-1}$ on C_1 and $z^{s-1} = e^{2\pi i(s-1)}t^{s-1}$ on C_3, then, as $\rho \to 0$,

$$\begin{aligned} I(s,x;A_j) &= \int_{\infty}^{0} \frac{t^{s-1}e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)}\ dt + \int_{0}^{\infty} \frac{e^{2\pi i(s-1)}t^{s-1}e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)}\ dt \\ &= (e^{2\pi i(s-1)}-1) \int_{0}^{\infty} \frac{t^{s-1}e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)}\, dt \\ &= (e^{2\pi i s}-1)\Gamma(s)\zeta(s,x;A_j). \end{aligned}$$

Now,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin\pi s} = \frac{2\pi i}{e^{\pi i s}-e^{-\pi i s}} = \frac{2\pi i\ e^{\pi i s}}{e^{2\pi i s}-1}.$$

Then,

$$I(s,x;A_j) = \frac{2\pi i\ e^{\pi i s}}{\Gamma(1-s)}\zeta(s,x;A_j) \quad \text{and,}$$

$$\zeta(s,x;A_j) = \frac{\Gamma(1-s)\ e^{\pi i s}}{2\pi i} I(s,x;A_j)$$
$$= \frac{\Gamma(1-s)\ e^{\pi i s}}{2\pi i} \int_C \frac{z^{s-1} e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\, dz, \quad Re(s) > r. \tag{3.5}$$

The integral $I(s,x;A_j)$ is obviously convergent, so the right side of Equation (3.5) is defined and meromorphic for all values of s. Therefore, Equation (3.5) provides the analytic continuation of $\zeta(s,x;A_j)$ to the whole s-plane. As shown above, $\zeta(s,x;A_j)$ is regular at $s > r$ and therefore at $s = r+1, r+2, \ldots$, and we know $\Gamma(1-s)$ is an analytic function for $Re(s) < 1$, so the only possible singularities are the poles at $s = 1, 2, \ldots, r$.

We first find a general formula for the residues of the zeta-function at these poles and then the particular formula for the residue at $s = r$. Let,

$$J(s,x;A_j) = \frac{1}{2\pi i} I(s,x;A_j) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\, dz$$

for the path C defined earlier. Then, $\int_{C_1} \ldots = -\int_{C_3} \ldots$ for $s = k$, $k = 1, 2, ...,$ and

$$J(k,x;A_j) = Res_{z=0} \frac{z^{k-1} e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}.$$

We now evaluate the residue at $k = r$. We have,

$$J(r,x;A_j) = \lim_{z\to 0} \frac{z^r}{\prod_{l=1}^{r}(a_{l,j}z + O(z^2))} = \frac{1}{\prod_{l=1}^{r} a_{l,j}}.$$

Using,

$$\Gamma(r-s) = \Gamma(r-1-s+1) = (r-s-1)\Gamma(r-1-s) = (r-s-1)\ldots(1-s)\Gamma(1-s),$$

we obtain,

$$\begin{aligned}\lim_{s\to r}(r-s)\Gamma(1-s) &= \lim_{s\to r}\frac{\Gamma(r-s+1)}{(r-s-1)\dots(1-s)}\\ &= \frac{1}{(-1)(-2)\dots(1-r)}\\ &= (-1)^{r-1}\frac{1}{(r-1)!}.\end{aligned}$$

Then,

$$\begin{aligned}Res_{s\to r}\zeta(s,x;A_j) &= \lim_{s\to r}(s-r)\Gamma(1-s)\ e^{2\pi is}J(r,x;A_j)\\ &= (-1)^{1+r-1+r}\frac{1}{(r-1)!}\frac{1}{\prod_{l=1}^{r}a_{l,j}},\end{aligned}$$

giving,

$$Res_{s\to r}\zeta(s,x;A_j) = \frac{1}{(r-1)!}\frac{1}{\prod_{l=1}^{r}a_{l,j}}. \tag{3.6}$$

Using Equation (3.5), we now calculate the special values of the zeta-function at non-positive integral values of $s=-k,\ k=0,1,2,.....$ We obtain,

$$\begin{aligned}\zeta(-k,x;A_j) &= (-1)^k\ \frac{\Gamma(k+1)}{2\pi i}\int_C \frac{z^{-k-1}e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\,dz\\ &= (-1)^k\ \ k!\ \ Res_{z=0}\ \ z^{-k-r-1}\left(\frac{z^r e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\right)\\ &= (-1)^k\ \ k!\ \ Res_{z=0}\ \ z^{-k-r-1}\sum_{n=0}^{\infty}B_n(\bar{w}_j;A_j)\frac{z^l}{n!},\end{aligned}$$

so that,

$$\zeta(-k,x;A_j) = (-1)^k\frac{k!}{(k+r)!}\ \ B_{k+r}(\bar{w}_j;A_j), \tag{3.7}$$

where $B_n(\bar{w}_j; A_j)$ is defined by the generating series:

$$\frac{t^r e^{\bar{w}_j t}}{\prod_{l=1}^{r}(e^{a_{l,j}t}-1)} = \sum_{n=0}^{\infty} \frac{B_n(\bar{w}_j; A_j)}{n!} t^n.$$

We also have,

$$\begin{aligned}
\zeta(s, x; A_j) &= \frac{\Gamma(1-s)\ e^{\pi i s}}{2\pi i} \int_C \frac{z^{s-1} e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\, dz, \\
&= \frac{\Gamma(1-s)\ e^{\pi i s}}{2\pi i} \int_C \frac{z^{s-1} e^{-w_j z}}{\prod_{l=1}^{r}(1-e^{-a_{l,j}z})}\, dz.
\end{aligned}$$

Then,

$$\begin{aligned}
\zeta(-k, x; A_j) &= (-1)^{k-k-1+r}\ k!\ \ Res_{z=0}\ \ (-z)^{-k-r-1} \left(\frac{(-z)^r e^{-w_j z}}{\prod_{l=1}^{r}(e^{-a_{l,j}z}-1)} \right) \\
&= (-1)^{r+1}\, k!\ \ Res_{z=0}\ \ (-z)^{-k-r-1} \sum_{n=0}^{\infty} B_n(w_j; A_j) \frac{(-z)^n}{n!},
\end{aligned}$$

giving,

$$\zeta(-k, x; A_j) = (-1)^r \frac{k!}{(k+r)!}\ \ B_{k+r}(w_j; A_j). \tag{3.8}$$

□

Remark 3. *The final result of this theorem allows us to generalize the well-known result $B_k(1-x) = (-1)^k B_k(x)$ to obtain*

$$B_k(xA_j; A_j) = (-1)^k B_k((\boldsymbol{1}-x)A_j; A_j) \tag{3.9}$$

where $\boldsymbol{1} = (\underbrace{1, \ldots, 1}_{r})$.

Remark 4. *The results obtained above show the special values of the zeta function are polynomials in w_j. This is the approach used by Stark. Shintani [56] on the other hand, obtained results which overtly show the special values are polynomials in* *the components x_i of the vector x. Such results can also be obtained here as follows. From Equation (**??**) we have,*

$$\zeta(-k,x;A_j) = (-1)^k \ k! \ Res_{z=0} \ z^{-k-r-1}\left(\frac{z^r e^{\bar{w}_j z}}{\prod_{l=1}^{r}(e^{a_{l,j}z}-1)}\right)$$

We can write the final bracketed term on the right side as

$$\left(\frac{ze^{(1-x_1)a_{1,j}z}}{e^{a_{1,j}z}-1}\right)\cdots\left(\frac{ze^{(1-x_r)a_{r,j}z}}{e^{a_{r,j}z}-1}\right)$$
$$=\left(\sum_{n_1=0}^{\infty}\frac{B_{n_1}(1-x_1)}{n_1!}(a_{1,j}z)^{n_1}\right)\cdots\left(\sum_{n_r=0}^{\infty}\frac{B_{n_r}(1-x_1)}{n_r!}(a_{r,j}z)^{n_1}\right). \quad (3.10)$$

It is then easy to see $\zeta(-k,x;A_j)$ is a polynomial in each x_i. In order to write the actual polynomial we need to separate the product of the infinite sums into the sum of r^2 products thus:

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \ = \ (n_1=0+\sum_{n_1=1}^{\infty})\ldots(n_r=0+\sum_{n_r=1}^{\infty}).$$

To satisfy the requirement $Res_{z=0} \ (-z)^{-k-r-1}$, the product $\sum_{n_1=1}^{\infty}\ldots\sum_{n_1=1}^{\infty}$ is easily written as

$$\sum_{n+1+\ldots+n_r=k+r}\prod_{i=1}^{r}\frac{B_{n_i}(1-x_i)}{n_i!}ai,j^{n_i}.$$

The writing of the other r^2-1 terms requires the more complicated notation given by Shintani [56], although again, it is easy to see each term is a polynomial in x_i. As an example, and for future use, the full expression for the $r=2$, $n=1$ case is given in the following Corollary.

Corollary 5. *For $A = A_{2,1}^{1,1}$, the value of $\zeta(s,x;A_1)$ at $s=-m$, $m=0,1,2,\ldots$, is given by*

$$\zeta(-m,x;A_1) = (-1)^m m!(R_1+R_2+R_3) \quad (3.11)$$

where,

$$R_1 = \frac{B_{m+2}(x_2)}{(m+2)!} \frac{a_{2,j}^m}{a_{1,j}},$$

$$R_2 = \frac{B_{m+2}(x_1)}{(m+2)!} \frac{a_{1,j}^m}{a_{2,j}},$$

$$R_3 = \sum_{\substack{k+l=m+2 \\ k,l \geq 1}} \frac{B_k(x_1)}{k!} \frac{B_l(x_2)}{l!} a_{1,j}^{k-1} a_{2,j}^{l-1}.$$

(Here, $B_j(y)$ is the usual Bernoulli polynomial defined by the generating function $\frac{te^{ut}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(u)}{k!} t^k$.)

Proof. Putting $r = 2$ and $s = -m$, in Equation (3.5), we arrive at

$$\begin{aligned}
\zeta(-m, x; A_1) &= (-1)^m \frac{\Gamma(m+1)}{2\pi i} \int_C \frac{z^{-m-1} e^{\bar{w}_j z}}{\prod_{l=1}^2 (e^{a_{l,j} z} - 1)} dz \\
&= (-1)^m \frac{\Gamma(m+1)}{2\pi i} \int_C z^{-m-1} \prod_{l=1}^{2} \frac{e^{(1-x_l) a_{1,l} z}}{(e^{a_{l,j} z} - 1)} dz \\
&= \text{coefficient of } z^m \text{ in the Laurent expansion at the origin of} \\
&\qquad (-1)^m m! \times \frac{e^{(1-x_1) a_{1,j} z}}{e^{a_{1,j} z} - 1} \times \frac{e^{(1-x_2) a_{2,j} z}}{e^{a_{2,j} z} - 1}.
\end{aligned}$$

Now, using the generating function, $\frac{te^{ut}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(u)}{k!} t^k$,, we have,

$$\begin{aligned}
\frac{e^{(1-x_1) a_{1,j} z}}{e^{a_{1,j} z} - 1} \times \frac{e^{(1-x_2) a_{2,j} z}}{e^{a_{2,j} z} - 1} &= \sum_{k=0}^{\infty} \frac{B_k(1-x_1)}{k!} (z a_{1,j})^{k-1} \times \sum_{l=0}^{\infty} \frac{B_l(1-x_2)}{l!} (z a_{2,j})^{l-1} \\
&\qquad \text{(splitting } \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \text{ into } \big(k = 0 \ + \ \sum_{k=1}^{\infty}\big)\big(l = 0 \ + \ \sum_{l=1}^{\infty}\big)) \\
&= R_1 + R_2 + R_3 + R_4,
\end{aligned}$$

where,

$$R_1 = \frac{B_0(1-x_1)}{(z a_{1,j})} \times \sum_{l=1}^{\infty} \frac{B_l(1-x_2)}{l!} ((z a_{2,j}))^{l-1}$$

for which the coefficient of z^m (requiring $l = m + 2$) is

$$B_0(1 - x_1)\frac{B_{m+2}(1 - x_2)}{(m + 2)!}a_{1,j}^{-1}\, a_{2,j}^{m}.$$

Similarly, the coefficient of z^{m-1} in R_2 is

$$B_0(1 - x_2)\frac{B_{m+2}(1 - x_1)}{(m + 2)!}a_{2,j}^{-1}\, a_{1,j}^{m},$$

and the coefficient of z^{m-1} in R_3 is

$$\sum_{\substack{k+l=m+2\\ k,l\geq 1}} \frac{B_k(1 - x_1)}{k!}\,\frac{B_l(1 - x_2)}{l!}\, a_{1,j}^{k-1}\, a_{2,j}^{l-1}.$$

Finally, R_4, with $k = l = 0$, cannot have a term in z^m. $\square$

Remark 6. *Since* $B_k(1 - x_j) = (-1)^k B_k(x_j)$, $\zeta(-m, x; A_1)$ *is a polynomial in each* x_j *of degree* $m + 2$.

For future use we note the following corollary.

Corollary 7. *In the* $r = 2$ *case, at* $m = 0$, *we have the results:*

$$\sum_{j=1}^{2} \zeta(0, x; A_j) = \frac{B_2(x_2)}{2}\Big[\frac{a_{2,1}}{a_{1,1}} + \frac{a_{2,2}}{a_{1,2}}\Big] + 2\, B_1(x_1)\, B_1(x_2) + \frac{B_2(x_1)}{2}\Big[\frac{a_{1,1}}{a_{2,1}} + \frac{a_{1,2}}{a_{2,2}}\Big]. \quad (3.12)$$

$$\sum_{j=1}^{2} \zeta(0, 0; A_j) = \frac{1}{12}\Big[\frac{a_{1,1}}{a_{2,1}} + \frac{a_{2,1}}{a_{1,1}} + \frac{a_{1,2}}{a_{2,2}} + \frac{a_{2,2}}{a_{1,2}}\Big] + \frac{1}{2}. \quad (3.13)$$

3.2.2 Generalized Bernoulli functions

Our next goal is to express the generalized Bernoulli functions $B_n(w_j; A_j)$ in terms of the first order, or usual, rescaled Bernoulli polynomials $B_n(x; f)$ with generating function $\frac{te^{xt}}{e^{ft}-1} = \sum_{n=0}^{\infty} \frac{B_n(x;f)}{n!} t^n$, and then as polynomials with coefficients containing Bernoulli numbers. Both the generalized and first order Bernoulli functions have been introduced because they will be the functions we require for the p-adic analysis we wish to perform. In particular, we show that for fixed A_j, the special values $\zeta(-k, x; A_j)$, $k = 0, 1, 2, \ldots$, are simply polynomials in each element x_i of the vector $x = (x_1, \ldots, x_r)$. We first note that the first order, rescaled Bernoulli polynomials may be expressed in terms of the usual Bernoulli polynomials.

Lemma 8. *We have:*

$$B_n(x; a) = a^{n-1} B_n(\frac{x}{a}). \tag{3.14}$$

Proof.

$$\sum_{n=0}^{\infty} \frac{B_n(x;a)}{n!} t^n = \frac{te^{xt}}{e^{at}-1} = \frac{1}{a} \frac{(at)e^{(\frac{x}{a})(ft)}}{e^{at}-1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{B_n(\frac{x}{a})}{n!} (at)^n$$

□

We now show the generalized Bernoulli polynomials, $B_n(w_j; A_j)$ are polynomials in w_j, and therefore in each component of x, of degree n.

Lemma 9. $B_n(w_j; A_j)$ *is the following polynomial in* w_j*:*

$$B_n(w_j; A_j) = n! \sum_{t=0}^{n} \binom{n}{t} w_j^{n-t} \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left(\frac{B_{n_i}}{n_i!} a_{i,j}^{n_i-1} \right) \tag{3.15}$$

where the n_i *are positive integers less than* n *or* 0.

Proof. $B_n(w_j; A_j)$ is generated by:

$$\frac{y^r e^{w_j y}}{\prod_{i=1}^r (e^{a_{i,j} y} - 1)} = \sum_{n=0}^{\infty} \frac{B_n(w_j; A_j)}{n!} y^n$$

and $B_n(0; A_j)$ is generated by:

$$\frac{y^r}{\prod_{i=1}^r (e^{a_{i,j} y} - 1)} = \sum_{n=0}^{\infty} \frac{B_n(0; A_j)}{n!} y^n.$$

Then,

$$\sum_{n=0}^{\infty} \frac{B_n(w_j; A_j)}{n!} y^n = e^{w_j y} \sum_{n=0}^{\infty} \frac{B_n(0; A_j)}{n!} y^n = \sum_{k=0}^{\infty} \frac{(w_j y)^k}{k!} \sum_{n=0}^{\infty} \frac{B_n(0; A_j)}{n!} y^n$$

$$\begin{aligned} \implies \frac{B_n(w_j; A_j)}{n!} &= \frac{w_j^n}{n!\, 0!} B_0(0; A_j) + \frac{w_j^{n-1}}{(n-1)!\, n!} B_n(0; A_j)) + \ldots + \frac{w_j^0}{0!\, n!} B_n(0; A_j)), \\ &= \sum_{t=0}^{n} \binom{n}{t} B_t(0; A_j) w_j^{n-t}. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n(0; A_j)}{n!} y^n &= \frac{y^r}{\prod_{i=1}^r (e^{a_{i,j} y} - 1)} = \frac{y}{e^{a_{1,j} y} - 1} \frac{y}{e^{a_{2,j} y} - 1} \cdots \frac{y}{e^{a_{r,j} y} - 1} \\ &= \prod_{i=1}^{r} \sum_{k=0}^{\infty} \frac{B_k(0; a_{i,j})}{k!} y^k \end{aligned}$$

$$\implies \quad B_n(0; A_j) = n! \sum_{n_1 + \ldots + n_r = n} \frac{B_{n_1}(0; a_{1,j})}{n_1!} \frac{B_{n_2}(0; a_{2,j})}{n_2!} \cdots \frac{B_{n_r}(0; a_{r,j})}{n_r!}$$

We conclude:

$$B_n(w_j; A_j) = n! \sum_{t=0}^{n} \binom{n}{t} w_j^{n-t} \sum_{n_1 + \ldots + n_r = t} \frac{B_{n_1}(0; a_{1,j})}{t_1!} \frac{B_{n_2}(0; a_{2,j})}{n_2!} \cdots \frac{B_{n_r}(0; a_{r,j})}{n_r!}$$

$$= n! \sum_{t=0}^{n} \binom{n}{t} w_j^{n-t} \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left(\frac{B_{n_i}}{n_i!} a_{i,j}^{n_i-1} \right)$$

□

Remark 10. We now recall the Von Staudt-Clausen theorem ([69], page 55):

Let n be even and positive and B_n be the n-th Bernoulli number. Then

$$B_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z}$$

where the sum is over all primes p such that $p-1$ divides n. Consequently, pB_n is p-integral for all n and all p.

We note that, due to the relations between the Bernoulli polynomials and Bernoulli numbers given by Lemma (9), this theorem implies $p^n B_n(w_j; A_j)$ has p-integral coefficients.

3.2.3 Corollaries

We have the following corollaries to Theorem (2).

Corollary 11. *In the $r = 1$, $n = 1$ case we have for the re-scaled Hurwitz zeta function:*

$$\zeta(s, x; A) = \sum_{n=0}^{\infty} (na_{1,1} + x_1 a_{1,1})^{-s} = \sum_{n=0}^{\infty} (na_{1,1} + w_j)^{-s}$$

that $\zeta(s, x; A)$ analytically continues to the whole complex s-plane, except for a first order pole at $s = 1$ of residue $\frac{1}{a_{1,1}}$. We have the values at $s = -k$, k a non-negative *integer,*

$$\zeta(-k, x; A) = -\frac{B_{k+1}(a_{1,1}x_1; a_{1,1})}{k+1} = (-1)^k \frac{B_{k+1}((1-x_1)a_{1,1}; a_{1,1})}{k+1} \tag{3.16}$$

where $B_{k+1}(a; f)$ is a rescaled Bernoulli polynomial with generating function

$$\frac{te^{at}}{e^{ft}-1} = \sum_{j=0}^{\infty} \frac{B_j(a; f)}{j!} t^j,$$

and, we can express $B_k(a; f)$ in terms of the usual Bernoulli polynomials and then in terms of Bernoulli numbers:

$$B_k(a; f) = f^{k-1} B_k(\frac{a}{f}) = f^{k-1} \sum_{j=0}^{k} \binom{k}{j} \; B_j \; (\frac{a}{f})^{k-j}$$

where $B_j(x)$ is the j-th Bernoulli polynomial and B_j is the j-th binomial number.

Corollary 12. *In the case of two summation variables ($r = 2$) and one linear form ($n = 1$), we have the Barnes [3] double ζ-function:*

$$\begin{aligned} \zeta(s, x; A_j) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} (m_1 a_{1,j} + m_2 a_{2,j} + x_1 a_{1,j} + x_2 a_{2,j})^{-s}, \;\; Re(s) > 2 \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} (m_1 a_{1,j} + m_2 a_{2,j} + w_j)^{-s} \end{aligned}$$

where $x = (x_1, x_2)$, $A_j = (a_{1,j}, a_{2,j})^t$ and $w_j = xA^{1,j}_{2,j}$.

Then, $\zeta(s, x; A_j)$ analytically continues to the whole complex s-plane, except for first order poles at $s = 1$ and $s = 2$, the latter with residue $\frac{1}{a_{1,j}a_{2,j}}$.

The special values at $s = -k$, k a non-negative integer, are given by:

$$\zeta(-k, x; A_j) = \frac{1}{(k+1)(k+2)} B_{k+2}(w_j; A_j), \tag{3.17}$$

where $B_k(w_j; A_j)$ is a polynomial of degree k with generating function

$$\frac{t^2 e^{w_j t}}{(e^{a_{1,j}t} - 1)(e^{a_{2,j}t} - 1)} = \sum_{j=0}^{\infty} \frac{B_k(w_j; A_j)}{k!} \; t^k,$$

and, we can express $B_k(w_j; A_j)$ in terms of the usual (rescaled) Bernoulli polynomials:

$$\begin{aligned} B_k(w_j; A_j) &= \sum_{l=0}^{k} \binom{k}{l} \; l! \sum_{l_1+l_2=l} \frac{B_{l_1}(0, a_{1,j})}{l_1!} \; \frac{B_{l_2}(0, a_{2,j})}{l_2!} \; w_j^{k-l} \\ &= \sum_{l=0}^{k} \binom{k}{l} \sum_{r=0}^{l} \binom{l}{r} B_r(0, a_{1,j}) \; B_{l-r}(0, a_{2,j}) \; w_j^{k-l}. \end{aligned}$$

Remark 13. *Equation* (3.17) *corrects a mistake on page 180 of [66].*

3.3 Complex Zeta functions with n linear forms

We return to the definition of a generalized ζ-function with r summation variables and n linear forms as given by Definition (1) in Section (3.1).

$$\begin{aligned} \zeta(s, x; A) = \zeta(s, x; A^{1,1}_{r,n}) &= \sum_{m \in \mathbb{Z}_0^r} \prod_{j=1}^{n} L_j(m+x)^{-s} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{j=1}^{n} \left(\sum_{i=1}^{r} (m_i + x_i) a_{i,j} \right)^{-s} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{j=1}^{n} \left(w_j + \sum_{i=1}^{r} m_i a_{i,j} \right)^{-s}. \end{aligned}$$

We again want to show these functions have analytic continuation and to find their values at non-positive integers. We can no longer simply apply the method of Theorem

(2), but require the innovations due to Shintani.

3.3.1 Analytic continuation and values at negative integers

Theorem 14. *(Shintani [56]) $\zeta(s,x;A)$ is absolutely convergent if $Re(s) > \frac{r}{n}$ and has an analytic continuation to a meromorphic function in the whole complex plane. Moreover, if we put $1-x = (1-x_1,\dots,1-x_r)$ the value at $s = 1-m,\ \ m = 1,2,\dots$ is*

$$\zeta(1-m,x;A) = (-1)^{(n-1)(m-1)}((m-1)!)^n \sum_{k=1}^{n} \frac{B_m(1-x,A)^{(k)}}{n} \tag{3.18}$$

where $B_m(1-x,A)^{(k)}$ is the coefficient of $u^{n(m-1)}\big(\prod_{l\neq k} y_l\big)^{m-1}$ in the Laurent expansion at the origin of

$$\prod_{i=1}^{r} \frac{e^{u(1-x_i)\sum_{j=1}^{n} a_{i,j}y_j}}{e^{u\sum_{j=1}^{n} a_{i,j}y_j}-1}\Bigg|_{y_k=1}$$

.

Proof. With $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$, if $Re(s) > \frac{r}{n}$, consider,

$$\Gamma(s)^n \prod_{j=1}^{n} \big[\sum_{i=1}^{r}(m_i a_{i,j} + x_i a_{i,j})\big]^{-s}$$
$$= \int_0^\infty e^{-t_1}\big[\sum_{i=1}^{r}(m_i a_{i,1} + x_i a_{i,1})\big]^{-s} t_1^{s-1}dt_1 \dots \int_0^\infty e^{-t_n}\big[\sum_{i=1}^{r}(m_i a_{i,n} + x_i a_{i,n})\big]^{-s} t_n^{s-1}dt_n$$

(Put, $t_j \mapsto \sum_{i=1}^{r}(m_i a_{i,j} + x_i a_{i,j})t_j$)

$$= \int_0^\infty e^{-\sum_{i=1}^{r}(m_i a_{i,1}+x_i a_{i,1})t_1} t_1^{s-1}dt_1 \dots\dots \int_0^\infty e^{-\sum_{i=1}^{r}(m_i a_{i,n}+x_i a_{i,n})t_n} t_n^{s-1}dt_n.$$

Then, $\Gamma(s)^n \zeta(s,x,;A)$

$$= \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{r} e^{-m_i \sum_{j=1}^{n} a_{i,j}t_j} \prod_{i=1}^{r} e^{-x_i \sum_{j=1}^{n} a_{i,j}t_j} (t_1 \dots t_n)^{s-1} dt_1 \dots dt_n$$
$$= \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{r} \frac{e^{(1-x_i)\sum_{j=1}^{n} a_{i,j}t_j}}{e^{\sum_{j=1}^{n} a_{i,j}t_j}-1} (t_1 \dots t_n)^{s-1} dt_1 \dots dt_n.$$

Define $D_k \subseteq \mathbb{R},\ \ k = 1, 2, \dots, n$ by,

$$D_k = \{t \in \mathbb{R}^n | 0 \le t_l \le t_k,\ \ l = 1, \dots, k-1, k+1, \dots, n\}$$

and let,

$$g(t) = \prod_{i=1}^{r} \frac{e^{(1-x_i)\sum_{j=1}^{n} a_{i,j} t_j}}{e^{\sum_{j=1}^{n} a_{i,j} t_j} - 1}.$$

Then

$$\begin{aligned} \zeta(s, x; A) &= \Gamma(s)^{-n} \int_0^\infty \dots \int_0^\infty g(t)(t_1 \dots t_n)^{s-1} dt_1 \dots dt_n \\ &= \Gamma(s)^{-n} \sum_{k=1}^{n} \int_{D_k} g(t)(t_1 \dots t_n)^{s-1} dt_1 \dots dt_n. \end{aligned}$$

In D_k we make the change of variables $t = u(y) = u(y_1, \dots, y_n)$ where $0 < u$, $0 \le y_l \le 1$ for $l \ne k$ and $y_k = 1$. The determinant of the Jacobian for this transformation is u^{n-1}. Let,

$$\begin{aligned} \zeta^{(k)}(s, x; A) &= \Gamma(s)^{-n} \int_{D_k} g(t)(t_1 \dots t_n)^{s-1} dt_1 \dots dt_n \\ &= \Gamma(s)^{-n} \int_0^\infty du \int_0^1 \dots \int_0^1 g(u(y)) u^{ns-1} \big(\prod_{l \ne k} y_l\big)^{s-1} \big(\prod_{l \ne k} dy_l\big) \end{aligned}$$

For a positive number ρ, denote by $I_\rho(1)$ and $I_\rho(\infty)$ the integral paths in $\mathbb{C}$ consisting of the intervals $[1, \rho]$ and $[\infty, \rho]$, counterclockwise around a circle of radius ρ and the intervals $[\rho, 1]$ and $[\rho, \infty]$ respectively. We have $z = t$ on $[1, \rho]$ and $[\infty, \rho]$ and $z = te^{2\pi i}$

on $[\rho, 1]$ and $[\rho, \infty]$.. Given the n linear forms have positive coefficients, we evaluate

$$\int_{I_\rho(\infty)} dw \int_{I_\rho(1)} g(w(z)) w^{ns-1} \big(\prod_{l \neq k} z_l\big)^{s-1} \big(\prod_{l \neq k} dz_l\big).$$

as $\rho \to 0$, obtaining, as in Theorem (2), a factor of $(e^{2\pi i s} - 1)$ from each of the $n-1$ inner integrals due to the terms z_l^{s-1} , but then a factor of $(e^{2n\pi i s} - 1)$ due to the term w^{ns-1}.

We then have:

$$\begin{aligned}
&\zeta^{(k)}(s, x; A) \\
&= \frac{\Gamma(s)^{-n}}{(e^{2n\pi i s} - 1)(e^{2\pi i s} - 1)^{n-1}} \int_{I_\rho(\infty)} du \int_{I_\rho(1)} g(u(y)) u^{ns-1} \big(\prod_{l \neq k} y_l\big)^{s-1} \big(\prod_{l \neq k} dy_l\big).
\end{aligned}$$

As a function of s, this integral is meromorphic in $\mathbb{C}$. Moreover,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin\ \pi s} = \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}} = \frac{2\pi i\ \ e^{\pi i s}}{e^{2\pi i s} - 1}$$

$$\Longrightarrow \ \frac{\Gamma(s)^{-n}}{(e^{2n\pi i s} - 1)(e^{2\pi i s} - 1)^{n-1}} = (2\pi i)^{-n} \Gamma(1-s)^n e^{n\pi i s} \frac{e^{2\pi i s} - 1}{e^{2n\pi i s} - 1}.$$

Now, at $s = 1 - m,\ m = 1, 2, \ldots,\ \ e^{n\pi i s} = (-1)^{n(m-1)}$ and $\frac{e^{2\pi i s} - 1}{e^{2n\pi i s} - 1} = \frac{1}{n}$, so,

$$\begin{aligned}
&\zeta^{(k)}(1-m, x; A) \\
&= (-1)^{n(m-1)} (2\pi i)^{-n} \frac{\Gamma(m)^n}{n} \int_{I_\rho(\infty)} du \int_{I_\rho(1)} g(u(y)) u^{n(1-m)-1} \big(\prod_{l \neq k} y_l\big)^{-m} \big(\prod_{l \neq k} dy_l\big) \\
&= (-1)^{n(m-1)} (2\pi i)^{-n} \frac{\Gamma(m)^n}{n} (2\pi i)^n \times \text{coefficient of} \ \ u^{n(m-1)} \big(\prod_{l \neq k} y_l\big)^{m-1}
\end{aligned}$$

in the Laurent expansion at the origin of,

$$g(uy_1, uy_2, \ldots, uy_{k-1}, uy_{k+1}, \ldots, uy_n) = \prod_{i=1}^{r} \frac{e^{u(1-x_i)\sum_{j=1}^{n} a_{i,j}y_j}}{e^{u\sum_{j=1}^{n} a_{i,j}y_j} - 1}\Bigg|_{y_k=1}.$$

Then,

$$\zeta(1-m, x; A) = (-1)^{n(m-1)}((m-1)!)^n \sum_{k=1}^{n} \frac{B_m(1-x, A)^{(k)}}{n}$$

where $B_m(1-x, A)^{(k)}$ is the coefficient of $u^{n(m-1)}\big(\prod_{l\neq k} y_l\big)^{m-1}$ in the Laurent expansion at the origin of $\prod_{i=1}^{r} \frac{e^{u(1-x_i)\sum_{j=1}^{n} a_{i,j}y_j}}{e^{u\sum_{j=1}^{n} a_{i,j}y_j}-1}\Big|_{y_k=1}$. □

Corollary 15. *(Shintani[1]) The value of the Dirichlet series* $\zeta(s, x; A_{r,n}^{1,1})$ *at* $s = 1-m$, $(m = 1, 2, \ldots)$ *is equal to* $(-1)^r m^{-n} B_m(x, A_{r,n}^{1,1})$ *where*

$$\frac{B_m(x, A_{r,n}^{1,1})}{(m!)^n} = \sum_{p} \frac{B_{p_1}(x_1)\ldots B_{p_r(x_r)}}{p_1!p_2!\ldots p_r!} C(A_{r,n}^{1,1}, p) + \frac{1}{n}\sum_{S}\sum_{q}\Big(\prod_{j\in S}\frac{B_{q(j)}(x_j)}{q(j)!}\Big)\sum_{k=1}^{n} c(S, q, A_{r,n}^{1,1})^{(k)}$$

and,

(1) $B_k(t)$ *is the usual* k*-th Bernoulli polynomial,*

(2) the summation with respect to p *is taken over all* r*-tuples of positive integers* $p = (p_1, p_2, \ldots, p_r)$ *which satisfy* $p_1 + p_2 + \ldots + p_r = n(m-1) + r$,

(3) $C(A_{r,n}^{1,1}, p)$ *is the coefficient of* $(t_1 \ldots t_n)^{m-1}$ *in the polynomial* $\prod_{j=1}^{r}(\sum_{k=1}^{n} a_{j,k}t_k)^{p_j-1}$,

(4) the summation with respect to S *is taken over all the proper and non-empty subsets of indices* $1, 2, \ldots, n$ *for each* S,

(5) the summation with respect to q *is over all the mappings from* S *to the set of positive integers which satisfy* $\sum_{j\in S} q(j) = n(m-1) + r$,

(6) $c(S, q, A_{r,n}^{1,1})$ *is the coefficient of* $(t_1 \ldots t_{k-1}t_{k+1} \ldots t_n)^{m-1}$ *in the Taylor expansion at the origin of the function* $\frac{\prod_{j\in S}(\sum_{k=1}^{n} a_{j,k}t_k)^{q_j-1}}{\prod_{j\notin S}(\sum_{k=1}^{n} a_{j,k}t_k)}\Big|_{t_k=1}$ *at the origin.*

Remark 16. *In the case of two linear forms and two summation variables, ($n = 2$, $r = 2$), if we begin the proof of Theorem* (14) *with (using the changed notation from x to w),*

$$\zeta(s, w; A) = \sum_{m \in \mathbb{Z}_0^r} \prod_{j=1}^{n} \left(w_j + \sum_{i=1}^{r} m_i a_{i,j} \right)^{-s},$$

we arrive at:

$\zeta^{(1)}(1-m, w; A)$ *equals the coefficient of* $u^{2(m-1)} y_2^{m-1}$ *in the Laurent expansion at the origin of*

$$\frac{\Gamma(m)^2}{2} e^{-(w_1 + w_2 y_2)u} \frac{e^{(a_{1,1}+a_{1,2}y_2)u}}{e^{(a_{1,1}+a_{1,2}y_2)u} - 1} \frac{e^{(a_{2,1}+a_{2,2}y_2)u}}{e^{(a_{2,1}+a_{2,2}y_2)u} - 1},$$

with a similar result for $\zeta^{(2)}(1-m, w; A)$. *This makes it explicit that* $\zeta(s, w; A)$ *is a polynomial in* w_1 *and* w_2.

The results we require for the case where the base field is a quadratic number field are as follows.

Corollary 17. *For* $A = A_{2,2}^{1,1}$, *the value of* $\zeta(s, x; A)$ *at* $s = 1 - m$, $m = 1, 2, \ldots$, *is given by*

$$\zeta(1-m, x; A) = \frac{((m-1)!)^2}{2} (R_1 + R_2 + R_3 + R_1' + R_2' + R_3') \tag{3.19}$$

where,

$$R_1 = \frac{B_{2m}(x_2)}{(2m)!} \sum_{\substack{k+j=m-1 \\ k,j \geq 0}} (-1)^k \binom{2m-1}{j} a_{1,1}^{-k-1}\, a_{1,2}^{k}\, a_{2,1}^{2m-j-1} a_{2,2}^{j}$$

$$R_2 = \frac{B_{2m}(x_1)}{(2m)!} \sum_{\substack{k+j=m-1 \\ k,j \geq 0}} (-1)^k \binom{2m-1}{j} a_{2,1}^{-k-1}\, a_{2,2}^{k}\, a_{1,1}^{2m-j-1} a_{1,2}^{j}$$

$$R_3 = \sum_{\substack{k+l=2m \\ k,l \geq 1}} \frac{B_k(x_1)}{k!} \frac{B_l(x_2)}{l!} \sum_{\substack{i+j=m-1 \\ i,j \geq 0}} \binom{k-1}{j} \binom{l-1}{i} a_{1,1}^{k-j-1}\, a_{1,2}^{j}\, a_{2,1}^{l-i-1}\, a_{2,2}^{i}$$

and, for $k = 1, 2, 3$, we obtain R'_k from R_k by interchanging $a_{1,j}$ with $a_{2,j}$ for $j = 1, 2$. (Here, $B_j(y)$ is the usual Bernoulli polynomial defined by the generating function $\frac{te^{ut}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(u)}{k!} t^k$.)

Proof. We have

$$\zeta(1-m, x; A) = \zeta(1-m, x; A_{2,2}^{1,1}) = ((m-1)!)^2 \sum_{k=1}^{2} \frac{B_m(1-x, A)^{(k)}}{2}$$

where $B_m(1-x, A)^{(k)}$ is the coefficient of $u^{2m-2} y_2^{m-1}$ in the Laurent expansion at the origin of

$$\prod_{i=1}^{2} \frac{e^{u(1-x_i)\sum_{j=1}^{2} a_{i,j} y_j}}{e^{u\sum_{j=1}^{2} a_{i,j} y_j} - 1}\Bigg|_{y_k=1}.$$

Write

$$\zeta(1-m, x; A) = \zeta^{(1)}(1-m, x; A) + \zeta^{(2)}(1-m, x; A), \quad \text{where,}$$

$\zeta^{(1)}(1-m, x; A) =$ coefficient of $u^{2m-2} y_2^{m-1}$ in the Laurent expansion at the origin of

$$\frac{((m-1)!)^2)}{2} \times \frac{e^{(1-x_1)(a_{1,1}+a_{1,2}y_2)u}}{e^{(a_{1,1}+a_{1,2}y_2)u} - 1} \times \frac{e^{(1-x_2)(a_{2,1}+a_{2,2}y_2)u}}{e^{(a_{2,1}+a_{2,2}y_2)u} - 1},$$

$\zeta^{(2)}(1-m, x; A) =$ coefficient of $u^{2m-2} y_1^{m-1}$ in the Laurent expansion at the origin of

$$\frac{((m-1)!)^2}{2} \times \frac{e^{(1-x_1)(a_{1,1}y_1+a_{1,2})u}}{e^{(a_{1,1}y_1+a_{1,2})u} - 1} \times \frac{e^{(1-x_2)(a_{2,1}y_1+a_{2,2})u}}{e^{(a_{2,1}y_1+a_{2,2})u} - 1}.$$

We need only consider $\zeta^{(1)}(1-m, x; A)$ since the expression for $\zeta^{(2)}(1-m, x; A)$ can be obtained from that result by simply interchanging $a_{1,1}$ with $a_{1,2}$ and $a_{2,1}$ with $a_{2,2}$. Now, using the generating function $\frac{te^{yt}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(y)}{k!} t^k$, we have,

$$\frac{e^{(1-x_1)(a_{1,1}+a_{1,2}y_2)u}}{e^{(a_{1,1}+a_{1,2}y_2)u}-1} \times \frac{e^{(1-x_2)(a_{2,1}+a_{2,2}y_2)u}}{e^{(a_{2,1}+a_{2,2}y_2)u}-1}$$
$$= \sum_{k=0}^{\infty} \frac{B_k(1-x_1)}{k!}(u(a_{1,1}+a_{1,2}y_2))^{k-1} \times \sum_{l=0}^{\infty} \frac{B_l(1-x_2)}{l!}(u(a_{2,1}+a_{2,2}y_2))^{l-1}$$
$$\text{(splitting } \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \text{into } \big(k=0 \ + \ \sum_{k=1}^{\infty}\big)\big(l=0 \ + \ \sum_{l=1}^{\infty}\big)\big)$$
$$= T_1 + T_2 + T_3 + T_4.$$

where

$$T_1 = \frac{B_0(1-x_1)}{u(a_{1,1}+a_{1,2}y_2)} \times \sum_{l=1}^{\infty} \frac{B_l(1-x_2)}{l!}(u(a_{2,1}+a_{2,2}y_2))^{l-1},$$

for which the coefficient of $u^{2m-2}y_2^{m-1}$ (requiring $l = 2m$) is,

$$B_0(1-x_1)\frac{B_{2m}(1-x_2)}{(2m)!} \times \text{coefficient of } y_2^{m-1} \text{ in } (a_{1,1}+a_{1,2}y_2)^{-1}(a_{2,1}+a_{2,2}y_2)^{2m-1}$$
$$= \frac{B_{2m}(x_2)}{(2m)!} \sum_{\substack{k+j=m-1 \\ k,j\geq 0}} (-1)^k \binom{2m-1}{j} a_{1,1}^{-k-1}\, a_{1,2}^{k}\, a_{2,1}^{2m-j-1} a_{2,2}^{j}.$$

Similarly, the coefficient of $u^{2m-2}y_2^{m-1}$ in T_2 is,

$$\frac{B_{2m}(x_1)}{(2m)!} \sum_{\substack{k+j=m-1 \\ k,j\geq 0}} (-1)^k \binom{2m-1}{j} a_{2,1}^{-k-1}\, a_{2,2}^{k}\, a_{1,1}^{2m-j-1} a_{1,2}^{j},$$

and the coefficient of $u^{2m-2}y_2^{m-1}$ in T_3 is,

$$\sum_{\substack{k+l=2m \\ k,l\geq 1}} \frac{B_k(x_1)}{k!}\,\frac{B_l(x_2)}{l!} \sum_{\substack{i+j=m-1 \\ i,j\geq 0}} \binom{k-1}{j}\binom{l-1}{i}\, a_{1,1}^{k-j-1}\, a_{1,2}^{j}\, a_{2,1}^{l-i-1}\, a_{2,2}^{i}.$$

Finally, R_4, with $k = l = 0$, cannot have a term in $u^{2m}y_2^m$. $\square$

Corollary 18. *For the $r = 2$, $n = 2$ case with $A = A^{1,1}_{2,2}$, we have the results at $m = 1$,*

$$\zeta(0, x; A) \; = \; \frac{B_2(x_2)}{2}\Big[\frac{a_{2,1}}{a_{1,1}} + \frac{a_{2,2}}{a_{1,2}}\Big] + 2\,B_1(x_1)\,B_1(x_2) + \frac{B_2(x_1)}{2}\Big[\frac{a_{1,2}}{a_{2,2}} + \frac{a_{1,1}}{a_{2,1}}\Big]. \quad (3.20)$$

$$\zeta(0, 0; A) = \frac{1}{24}\Big[\frac{a_{1,1}}{a_{2,1}} + \frac{a_{2,1}}{a_{1,1}} + \frac{a_{1,2}}{a_{2,2}} + \frac{a_{2,2}}{a_{1,2}}\Big] + \frac{1}{4}. \quad (3.21)$$

Using Corollary (7) we then have,

$$\zeta(0, x; A) = \frac{1}{2}\big[\zeta(0, x; A_1) + \zeta(0, x; A_2)\big],$$

$$\zeta(0, 0; A) = \frac{1}{2}\big[\zeta(0, 0; A_1) + \zeta(0, 0; A_2)\big],$$

and, in particular,

$$\zeta(0, x; A) - \zeta(0, 0; A) = \frac{1}{2}\sum_{j=1}^{2}\big[\zeta(0, x; A_j) - \zeta(0, 0; A_j)\big]. \quad (3.22)$$

4 p-adic zeta functions

4.1 The Interpolation Problem

In Section (3), we generalized results in classical analysis for zeta functions due to Riemann, Barnes and others. We want to perform p-adic analysis on the related p-adic zeta functions, $\zeta_p(s, x; A)$. The p-adic zeta functions can be constructed from the modified complex zeta functions by a process known as p-adic interpolation. We say a function on the positive integers or any other dense subset of $\mathbb{Z}_p$ can be p-adically interpolated if it first has a (uniformly) continuous extension to all of $\mathbb{Z}_p$. We note that"continuous" means, as in the real case, that whenever a sequence of p-adic integers x_n approaches x p-adically, $f(x_n)$ approaches $f(x)$ p-adically. Second, the modified complex function and the continuous p-adic function must agree on a set of negative integers that is dense in $\mathbb{Z}_p$. Such a continuous p-adic extension is then unique, because two continuous functions that agree on a dense subset are identical. The explicit construction of a p-adic (zeta) function which satisfies these two conditions is what is meant by"solving the interpolation problem".

To prove p-adic uniform continuity for such functions $f(x)$, we will show that, *Given any real number $\epsilon > 0$, there is a real number $\delta > 0$ such that for any x,$y \in \mathbb{Z}_p$,*

$$|x - y|_p < \delta \implies |f(x) - f(y)|_p < \epsilon.$$

We will use this in the form: There is a positive integer N such that,

$$|x - y|_p \leq \frac{1}{p^N} \implies |f(x) - f(y)|_p \leq \frac{1}{p^{N+1}}.$$

Let us first consider the case of complex zeta functions with two summation variables and one linear form. They are related to quadratic number fields. In the notation of Definition (1), let $A_j = (a_{1,j}, a_{2,j})^t$, $x = (x_1, x_2)$ and $w_j = xA_j$, where we assume the $a_{i,j}$'s and x_i's are positive numbers. For $j = 1, 2$, by Corollary (12), we know the complex zeta function

$$\zeta(s, x; A_j) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} (m_1 a_{1,j} + m_2 a_{2,j} + w_j)^{-s}, \quad Re(s) > 2$$

has an analytic continuation to the whole complex plane apart from simple poles at $s = 1$ and $s = 2$. We can evaluate the analytically continued function at non-positive integers as polynomials in the components of x or as polynomials in w_j. We suppose the $a_{i,j}$'s and x_i's are integers in a totally real quadratic number field F and that the chosen rational prime p splits in F. We first want to embed the elements of F into a p-adic field $\mathbb{Q}_p$.

4.2 Embedding Number Fields in $\mathbb{Q}_p$

In the case of a quadratic number field $F = \mathbb{Q}(\sqrt{d})$, we have the following known result:

Lemma 19. *The following are equivalent:*

(1) A prime p splits (completely) in a quadratic number field $F = \mathbb{Q}(\sqrt{d})$

(2) The Legendre symbol $\left(\frac{d}{p}\right) = +1$

(3) $F \hookrightarrow \mathbb{Q}_p$, that is, there is an embedding of F into $\mathbb{Q}_p$.

Proof. (1)$\Rightarrow$(2): Assume p splits in F. Let,

$$(p) = (p, a + b\sqrt{d})(p, a - b\sqrt{d}) = (p^2, p(a + b\sqrt{d}), p(a - b\sqrt{d}), a^2 - b^2 d).$$

Then we have: $a^2 - b^2 d \equiv 0 (mod\ p) \Rightarrow d \equiv \dfrac{a^2}{b^2} (mod\ p) \Rightarrow \left(\dfrac{d}{p}\right) = +1.$

(2)⇒(1): Assume $d \equiv a^2 \pmod p$. Then $p \not| a$ but $p | a^2 - d$. Consider,

$$\begin{aligned}(p, a+\sqrt{d})(p, a-\sqrt{d}) &= (p^2, p(a+\sqrt{d}), p(a-b\sqrt{d}), a^2-d) \\ &= (p)(p, a+\sqrt{d}, a-\sqrt{d}, \frac{a^2-d}{p}) \\ &= (p)(p, 2a, \frac{a^2-d}{p}) \\ &= (p) \quad \text{or } p \text{ splits in } F, \\ &\text{since } p \not| a \Rightarrow (p, 2a) = 1 \Rightarrow 1 \in (p, 2a, \frac{a^2-d}{p}).\end{aligned}$$

(2) ⇒ (3): This is a well-known result (see, for example, Katok [38], page 37) obtained by applying Hensel's lemma. *Let $f(x)$ be a polynomial whose coefficients are in Z_p. If there is an $\alpha_1 \in \mathbb{Z}_p$ such that $f(\alpha_1) \equiv 0(mod\ p)$ but $f'(\alpha_1) \not\equiv 0(mod\ p)$ then there is a unique $\alpha \in \mathbb{Z}_p$ such that $\alpha \equiv \alpha_1(mod\ p)$ and $f(\alpha) = 0$.*
Let $f(x) = x^2 - d$. Then $f'(x) = 2x$. If d is a quadratic residue, then $d \equiv d_0^2(mod\ p)$ for some $d_0 \in 1, 2, \ldots, p-1$. Hence $f(d_0) \equiv 0(mod\ p)$. But $f'(d_0) = 2d_0 \not\equiv 0(mod\ p)$ since $(d_0, p) = 1$, so the solution in Z_p exists by Hensel's lemma.
(3) ⇒ (2): Suppose $\sqrt{d} = d_0 + d_1 p + \ldots$. Then $d \equiv d_0^2(mod\ p) \implies \frac{d}{p} = +1$. $\square$

To return to the p-adic interpolation of a function related to $\zeta(s, x; A)$. We suppose the elements of A are all algebraic numbers in some quadratic field F and that some prime p splits in F. Then, by Lemma (19), F embeds in $\mathbb{Q}_p$. Accordingly, if we assume p does not divide any of the elements of A_j, then these elements are all integers in $\mathbb{Q}_p$. We do not have free choice of the prime p. By Lemma (19) we have an embedding of $F = \mathbb{Q}(\sqrt{d})$ into $\mathbb{Q}_p$ only if $d \equiv \square \pmod p$, so, given d, we must choose some p accordingly. Then p must split (completely) in F into two factors of the form $(a + b\sqrt{d})$ and $(a - b\sqrt{d})$. There will be two choices for the embedding:

$$\sqrt{d} = \begin{cases} a_0 + a_1 p + a_2 p^2 + \ldots, & 0 \leq a_i < p \\ b_0 + b_1 p + b_2 p^2 + \ldots, & 0 \leq b_i < p \end{cases}$$

where we must have $a_0 + b_0 = p$. We can take one of the embeddings to be $\sqrt{d}$ and the other to be $-\sqrt{d}$. We first choose either one of the two factors of p as $\mathfrak{p}$ and then we choose the embedding of $\sqrt{d}$ into $\mathbb{Q}_p$ so that the p-adic valuation of any element $\beta \in \mathfrak{p}$ is $|\beta|_p \leq \frac{1}{p} < 1$.

Example 20. *We can embed $\mathbb{Q}(\sqrt{5})$ into $\mathbb{Q}_{11}$ since $(\frac{5}{11}) = +1$. Now, as expected, 11 splits as $11 = (4 + \sqrt{5})(4 - \sqrt{5})$. Choose $\mathfrak{p} = (4 + \sqrt{5})$. Then, given the choice of embeddings*

$$\sqrt{5} = \begin{cases} 4 + a_1 11 + a_2 11^2 + \ldots, & 0 \leq a_i < 11 \\ 7 + b_1 11 + b_2 11^2 + \ldots, & 0 \leq b_i < 11 \end{cases}$$

we choose $\sqrt{5} = 7 + b_1 11 + b_2 11^2 + \ldots$ so that $4 + \sqrt{5} = c_1 11 + c_2 11^2 + \ldots$ and $|\beta|_{11} \leq \frac{1}{11} < 1$ for any $\beta \in \mathfrak{p}$.

4.3 p-adic zeta functions with one linear form

4.3.1 Definition

We want to define a p-adic zeta function which solves the interpolation problem for a regularized zeta function defined as follows.

Definition 21. *Let*

$$\zeta^*(s, x; A_j) = \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p} \nmid L_j(m+x)}}^{\infty} L_j(m + x)^{-s} \quad Re(s) > 2. \tag{4.1}$$

where $L = L_j(m+x) = m_1 a_{1,j} + m_2 a_{2,j} + w_j$*.*

Note that, in the remainder of this paper, whenever we define a zeta function, the ζ^* *notation always means that we are removing the terms divisible by a fixed prime* p *or a specified prime divisor* $\mathfrak{p}$ *of* p*.*

We suppose w_j, $a_{1,j}$ and $a_{2,j}$ are all integers in a quadratic field F and that a fixed prime p splits in F, say $p = \mathfrak{p}\mathfrak{p}'$ where if $\beta \in \mathfrak{p}$ and $\beta' \in \mathfrak{p}'$ then $|\beta|_p < 1$ and $|\beta'|_p = 1$.

We now define the term "p-adic limit" and then establish a pair of p-adic limits we will use in our calculations.

Definition 22. *Let* $\alpha \in \mathbb{Z}_p$*. By* $N \underset{p}{\rightarrow} \alpha$ *we mean* N *approaches* α *p-adically through values in* $\mathbb{Z}^+$ *in such a way that* $N \longrightarrow \infty$ *as well.*

For example, if

$$\alpha = -1 = (p-1) + (p-1)p + (p-1)p^2 + \dots$$

we could let N run through the sequence of partial sums

$$(p-1),\ \ (p-1)+(p-1)p,\ \ (p-1)+(p-1)p+(p-1)p^2,\ \ \dots.$$

We now choose two sequences of integers M_1 and M_2 defined as follows.

Definition 23. *For* $\alpha,\ \beta \in \mathbb{Z}_p$*, we choose sequences of positive integers* M_1 *and* M_2 *such that*

$$M_1 \underset{p}{\rightarrow} \alpha,\ \ M_2 \underset{p}{\rightarrow} \beta.$$

Alternatively, we can define these p-adic limits as follows. Let $T \in \mathbb{Z}^+$ *be such that* $T \rightarrow +\infty$*.*

Define $M_1 \to +\infty$ such that

$$|M_1 - \alpha|_p \leq p^{-T} \Leftrightarrow M_1 - \alpha = p^T q_1, \;\; p \nmid q_1, \;\; q_1 \in \mathbb{Z}_p.$$

Similarly, define $M_2 \to +\infty$ such that

$$|M_2 - \beta|_p \leq p^{-T} \Leftrightarrow M_2 - \beta = p^T q_2, \;\; p \nmid q_2, \;\; q_2 \in \mathbb{Z}_p.$$

We will later choose specific values for α and β, but, for the present, we can regard them as general p-adic integers.

We now want to define, the p-adic function

$$H(s, x; A_j) = \lim_{T \to \infty} \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p} \nmid L_j(m+x)}}^{M_2-1} (L_j(m+x))^{-s}, \;\; s \in \mathbb{Z}_p, \tag{4.2}$$

where, $L_j(m+x) = \sum_{i=1}^{2} (m_i + x_i) a_{i,j}$, $A_j = (a_{1,j}, a_{2,j})^t$, $x = (x_1, x_2)$ and $w_j = xA_j$ (as in Definition (1)).

We first need to show $H(s, x; A_j)$ is well-defined and that entails defining $L_j(m+x)^{-s}$ for $\mathfrak{p} \nmid L_j(m+x)$. Since x_i and $a_{i,j}$ are in $\mathbb{Z}_p$ for all i and j, we can simply consider n^{-s} where $n \in \mathbb{Z}_p$ and $p \nmid n$. So, we consider the function $f(k) = n^{-k}$ where k is a non-negative integer, $n \in \mathbb{Z}_p$ and $(n, p) = 1$. We follow the arguments on page 26 of Koblitz [1] and page 126 of Gouvêa [2] and claim:

Lemma 24. *$f(k)$ admits a p-adic interpolation, that is, $f(k)$ may be extended in a unique way from the non-negative integers to the p-adic integers so that the resulting function is a continuous function of a p-adic variable s with values in $\mathbb{Z}_p$.*

Proof. For $s \in \mathbb{Z}_p$, we define the p-adic function,

$$f_p(s) = \lim_{k \to -s} n^k$$

where $p \nmid n$ and $k \in \mathbb{Z}^+$ runs through a sequence of integers approaching $-s$. As noted in the introductory remarks to this section, to prove p-adic continuity for this function, it suffices to show that,

$$|k' - k|_p \leq \frac{1}{p^N} \implies |n^{-k} - n^{-k'}|_p \leq \frac{1}{p^{N+1}}.$$

Since $(p-1) \in \mathbb{Z}_p^{\times}$, we can write

$$-s = (p-1)s' = (p-1)(a_0 + a_1 p + \ldots),$$

for some $s' \in \mathbb{Z}_p$, $s' = a_0 + a_1 p + \ldots$.

Then we choose as sequences of integers k, $(p-1)$ times the partial sums of s', so that

$$k \underset{p}{\rightarrow} -s, \quad k \equiv 0 (mod\, p-1).$$

Accordingly, if we have $k = (p-1)k_1$ and $k' = (p-1)k_2$, where $k_1,\, k_2 \in \mathbb{Z}^+$, then,

$$|k' - k|_p = |(p-1)k_1 - (p-1)k_2|_p = |k_1 - k_2|_p \leq \frac{1}{p^N}.$$

We therefore have $k_1 - k_2 = k_3 p^N$, where $0 \leq k_3 < p^N$. Now, using Fermat's little theorem to put $n^{p-1} = 1 + mp$ for $p \nmid n$,

$$\begin{aligned} \left|n^k - n^{k'}\right|_p &= \left|(n^{p-1})^{k_1} - (n^{p-1})^{k_2}\right|_p \\ &= \left|(n^{p-1})^{k_1}\right|_p \left|1 - (n^{p-1})^{k_2 - k_1}\right|_p \end{aligned}$$

$$
\begin{aligned}
&= \left|1-(1+mp)^{k_3p^N}\right|_p \\
&= \left|(1+mp)^{k_3p^N-1}\right|_p \\
&= \left|k_3p^Nmp+\binom{k_3p^N}{2}(mp)^2+\dots\binom{k_3p^N}{k}(mp)^k+\dots\right|_p \\
&\leq max(|k_3p^Nmp|_p,|\binom{k_3p^N}{2}(mp)^2|_p,\dots,|\binom{k_3p^N}{j}(mp)^j|_p,\dots) \\
&\leq \left|p^{N+1}\right|_p \\
&\leq \frac{1}{p^{N+1}}
\end{aligned}
$$

□

In summary,

Definition 25. *For $s \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_p^\times$, we define n^{-s} as*

$$n^{-s} = \lim_{k \to -s} n^k$$

where k runs through a sequence of non-negative integers congruent to $0(mod\ \ p-1)$ and tending to $-s$ p-adically.

This allows us to investigate the values of $H(s, x; A_j)$ at integers divisible by $p-1$, including 0. These integers must be even.

Second, now that we know n^{-s} is well-defined, to show $H(s, x; A_j)$, is well-defined, we need to show convergence. We start with the elementary result that an infinite series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ converges if and only if $\lim_{n\to\infty} a_n = 0$. Since the terms on the right side of (4.2) are not divisible by p, the p-adic value of each of them is 1, so the infinite series, on first inspection, does not appear to converge. We can, however, show convergence if we group the terms. We start with the infinite series

$$\sum_{\substack{n=0 \\ p\nmid n+x}}^{\infty} (n+x)^{-s},$$

and group the terms as follows:

$$\sum_{\substack{n=0 \\ p\nmid n+x}}^{\infty} (n+x)^{-s} = \sum_{\substack{0\le n<p^r \\ p\nmid n+x}} (n+x)^{-s} + \sum_{\substack{p^r\le n<2\,p^r \\ p\nmid n+x}} (n+x)^{-s} + \dots$$

We will prove the following lemma not just for one linear form but, for future use, in the general case of m linear forms.

Lemma 26. *Suppose s, $x_i \in \mathbb{Z}_p$, $x = (x_1, \dots, x_k)$ and $p^a || s$. Then for $r \geq a+1$,*

$$\sum_{\substack{n (mod\, p^r) \\ p\nmid \prod_{i=1}^m (n+x_i)}} \prod_{i=1}^{m} (n+x_i)^{-s} \equiv \begin{cases} 0 (mod\, p^{r-a-1}), & \text{if } s \neq 0; \\ 0 (mod\, p^{r-1}), & \text{if } s = 0. \end{cases}$$

Proof. The case $s = 0$ is trivial.

$$\sum_{\substack{n (mod\, p^r) \\ p\nmid \prod_{i=1}^m (n+x_i)}} \prod_{i=1}^{m} (n+x_i)^{-s} \Big|_{s=0} = \underbrace{1+\ldots+1}_{p^r} - \left[\frac{p^r}{p}\right] = \frac{p^r}{2}(p^r-1) - p^{r-1} \equiv 0\,(mod\ p^{r-1}).$$

Suppose $s \neq 0$. Recall, for B_l the l-st binomial number, that

$$\sum_{n=0}^{N-1} n^j = \frac{1}{j+1} \sum_{l=0}^{j} B_l\, N^{j+1-l}.$$

Consider, for $t = (p-1)p^a\, u$, $u \in \mathbb{Z}^+$,

$$S = \sum_{n=0}^{p^r-1} \prod_{i=1}^{m} (n+x_i)^t$$

$$= \sum_{n=0}^{p^r-1} \sum_{j=0}^{mt} c_j \, n^j, \text{ say,}$$

$$= \sum_{j=0}^{mt} \frac{c_j}{j+1} \sum_{l=0}^{j} B_l \, p^{r(j+1-l)}$$

$$= \sum_{k=1}^{m} \frac{c_{kt-1}}{kt} B_{kt-1} \, p^r + \text{terms in } p^k, \ k \geq 2r$$

$$\equiv 0 (mod \ p^{r-a-1}) \ \text{ by the von Staudt-Clausen theorem.}$$

Now consider the terms in S that are divisible by p. Suppose $n + x_1$ is divisible by p. Let $x_1 = a_0 + p\,u$, $u \in \mathbb{Z}_p$. Then $p|(n + x_1)$ if $n = lp - a_0$, $1 \leq l \leq p^{r-1}$. Then we can write for values of y_i, $d_i \in \mathbb{Z}_p$,

$$\sum_{\substack{n (mod\, p^r) \\ p|\prod_{i=1}^{m}(n+x_i)}} \prod_{i=1}^{m} (n + x_i)^{-s} = \sum_{l=1}^{p^{r-1}} (lp + py_1)^t \prod_{i=2}^{m} (lp + y_i)^t$$

$$= p^t \sum_{l=1}^{p^{r-1}} \sum_{j=0}^{mt} d_j \, l^j$$

$$= p^t \Big[\sum_{l=1}^{p^{r-1}} d_0 + \sum_{j=1}^{mt} \sum_{l=1}^{p^{r-1}} d_j \, l^j \Big]$$

$$\equiv 0 (mod \ p^{r+a-1}).$$

Now put $-s = t$. □

Remark 27. *Stark [65] gives a different proof of the preceding lemma for the case of one linear form and one summation variable.*

We can now show that,

Lemma 28. *$H(s, x; A_j)$ converges and is continuous in both s, x_1 and x_2.*

Proof. To show convergence, note that if T increases to T', the group of extra terms added to

$$\sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2-1} L_j(m+x)^{-s}$$

is

$$\sum_{m_1=0}^{M_1'-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s} - \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2-1} L_j(m+x)^{-s}$$
$$= \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s} + \sum_{m_1=M_1}^{M_1'-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s}.$$

We claim the p-adic value of this group approaches zero as $T \to \infty$. Consider,

$$\sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s}.$$

Fix a value of m_1 and then consider

$$\sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} (w_j + \sum_{i=1}^{2} m_i a_{i,j})^{-s} = a_{2,j}^{-s} \sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid m_2+x}}^{M_2'-1} (m_2+y)^{-s}$$

where $y = \frac{w_j + m_1 a_{1,j}}{a_{2,j}} \in \mathbb{Z}_p$. Since the number of terms in this sum is $M_2' - M_2 \equiv 0 (mod\, p^T)$, we can apply Lemma (26). Then

$$|\sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s}|_p \to 0$$

as $T \to \infty$. The same argument applies to

$$\sum_{m_1=M_1}^{M_1'-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2'-1} L_j(m+x)^{-s}.$$

To show $H(s,x;A)$ is continuous in s, we let $|s_1 - s_2|_p < p^{-r}$ and $n = L_j(m+x)$, $n \in \mathbb{Z}_p$. Then,

$$|H(s_1,x;A_j) - H(s_2,x;A_j)|_p = \lim_{T\to\infty} \sum_{m_1=0}^{M_1-1} \sum_{m_2=0}^{M_2-1} |n^{-s_1} - n^{-s_2}|_p < p^{-r-1}$$

by the same method of proof as in Lemma (24).

To show $H(s,x;A)$ is continuous in x_1, we can choose two values x_{11} and x_{12} such that $|x_{11} - x_{12}|_p < p^{-r}$. Write $\bar{L}_j = m_1 a_{1,j} + m_2 a_{2,j} + x_2 a_{2j}$.
Then, with $k = 1, 2, \ldots$, $k \equiv 0 (mod\, p-1)$, for some value of m_1 and m_2,

$$|H(-k,(x_{11},\ldots);A_j) - H(-k,(x_{12},\ldots);A_j)|_p \le |(\bar{L}_j + x_{11}a_{1,j})^k - (\bar{L}_j + x_{12}a_{1,j})^k|_p < p^{-r}.$$

Let $-s = k$. Similarly, $H(s,x;A_j)$ is continuous in x_2. $\square$

We return to the problem of interpolating the (special) values at $s = -k$, $k = 0, 1, 2, \ldots$, $k \equiv 0(mod\ p-1)$ of the (regularized) complex zeta function defined by

$$\zeta^*(s,x;A_j) = \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1 a_{1,j} + m_2 a_{2,j} + w_j}}^{\infty} (m_1 a_{1,j} + m_2 a_{2,j} + w_j)^{-s}, \quad Re(s) > 2.$$

We need the following lemma for a zeta function, $\zeta(s,x;a) = \sum_{n=0}^{\infty} ((n+x)a)^{-s}$, with one summation variable and one linear form.

Lemma 29. *For even positive integers k, if $x \underset{p}{\to} 0$ then the value of $\zeta(-k, x; a) \underset{p}{\to} 0$.*

Proof. With $B_{k+1}(y)$ as the usual Bernoulli polynomial with generating function

$$\frac{te^{yt}}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j(y)}{j!} t^j,$$

we have by Equation (3.16),

$$\begin{aligned}
\zeta(-k, x; a) &= -\frac{B_{k+1}(ax; a)}{k+1} \\
&= -a^k \frac{B_{k+1}(x)}{k+1} \\
&= -\frac{a^k}{k+1} \sum_{l=0}^{k+1} \binom{k+1}{l} B_l \; x^{k+1-l} \\
&= -\frac{a^k}{k+1} \left[\sum_{l=0}^{k} \binom{k+1}{l} B_l \; x^{k+1-l} + B_{k+1} \right] \\
&= xP(x), \text{ for even } k,
\end{aligned}$$

where $P(x)$ is a polynomial in x. Then $\zeta(-k, x; a) \underset{p}{\to} 0$ as $x \underset{p}{\to} 0$. $\square$

In the light of what we need for the following theorem and its corollary, we now make our choices for the p-adic integers α and β of Definition (23).

Definition 30. *We define the sequence of positive integers M_1 such that*

$$M_1 \underset{p}{\to} -\frac{w_j}{a_{1,j}}.$$

Then for each m_1 such that $0 \leq m_1 < M_1$, we define the sequence of positive integers $M_2 = M_2(m_1)$ such that

$$M_2 \underset{p}{\to} -\frac{w_j + m_1 a_{1,j}}{a_{2,j}}.$$

Alternatively, we define these p-adic limits as:

Let $T \in \mathbb{Z}^+$ be such that $T \to +\infty$.

Choose $M_1 \to +\infty$ such that

$$|M_1 a_{1,j} + w_j|_p \leq p^{-T} \Leftrightarrow M_1 a_{1,j} + w_j = p^T q_1, \;\; p \nmid q_1, \;\; q_1 \in \mathbb{Z}_p.$$

For each m_1 such that $0 \leq m_1 < M_1$, choose $M_2 \to +\infty$ such that,

$$|M_2 a_{2,j} + m_1 a_{1,j} + w_j|_p \leq p^{-T} \Leftrightarrow M_2 a_{2,j} + m_1 a_{1,j} + w_j = p^T q_2, \;\; p \nmid q_2, \;\; q_2 \in \mathbb{Z}_p.$$

We have the following theorem for the difference equation of two ζ^* functions.

Theorem 31. *For k as above,*

$$\begin{aligned}\zeta^*(-k,(x_1,x_2);A_j) &- \zeta^*(-k,(x_1+M_1,x_2);A_j) \\ &= \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2-1} (m_1a_{1,j}+m_2a_{2,j}+w_j)^k + E(T) \end{aligned} \tag{4.3}$$

where $E(T) \to 0$ as $T \to \infty$.

Proof.

$$\begin{aligned}&\zeta^*(s,(x_1,x_2);A_j) - \zeta^*(s,(x_1+M_1,x_2);A_j) \\ &= \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j}+m_2a_{2,j}+w_j)^{-s} \\ &\qquad - \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+M_1a1,j+w_j}}^{\infty} (m_1a_{1,j}+m_2a_{2,j}+M_1a_{1,j}+w_j)^{-s}\end{aligned}$$

$$= \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + w_j)^{-s}$$

$$= \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2-1} (m_1a_{1,j} + m_2a_{2,j} + w_j)^{-s} + S,$$

where,

$$S = \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=M_2 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + w_j)^{-s}$$

$$= \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid m_1a_{1,j}+m_2a_{2,j}+M_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + M_2a_{2,j} + w_j)^{-s}$$

$$= \sum_{m_1=0}^{M_1-1} \sum_{m_2=0}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + M_2a_{2,j} + w_j)^{-s}$$

$$- \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\mid m_1a_{1,j}+m_2a_{2,j}+M_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + M_2a_{2,j} + w_j)^{-s}$$

Now the first double series is simply $\sum_{m_1=0}^{M_1-1} \zeta(s, \Omega; a_{2,j})$ where

$$\Omega = \Omega(m_1) = \frac{m_1a_{1,j} + M_2a_{2,j} + w_j}{a_{1,2}} \underset{p}{\rightarrow} 0$$

as $T \to \infty$. At $s = -k$, by Lemma (29), we have $\zeta(s, \Omega; a_{2,j}) \underset{p}{\rightarrow} 0$. In the second double series, the condition $\mathfrak{p}|m_1a_{1,j} + m_2a_{2,j} + M_2a_{2,j} + w_j$ implies $p|m_2$ by our choice of M_2. This allows us to write the second double series as $\sum_{m_1=0}^{M_1-1} \zeta(s, \Omega; pa_{2,j})$. Again, at $s = -k$, thanks to Lemma (29), we have $\zeta(s, \Omega; pa_{2,j}) \underset{p}{\rightarrow} 0$. $\square$

Corollary 32.

$$\zeta^*(-k, x; A_j) - \zeta^*(-k, 0; A_j) = H(-k, x; A_j).$$

Proof. We can write Equation (4.3) in the notation

$$\zeta^*(-k, w_j; A_j) - \zeta^*(-k, w_j + M_1 a_{1,j}; A_j) \\ = \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p} \nmid m_1 a_{1,j} + m_2 a_{2,j} + w_j}}^{M_2-1} (m_1 a_{1,j} + m_2 a_{2,j} + w_j)^k + E(T) \quad (4.4)$$

We need to show

$$\lim_{T \to \infty} \zeta^*(-k, w_j + M_1 a_{1,j}; A_j) = \zeta^*(-k, 0; A_j),$$

that is, we can pass to the limit for M_1 inside ζ^*. We first note that M_1 passing to the limit means we have $T \to \infty$ so that the error term $E(T)$ in Equation (4.3) is removed. Let $\mathfrak{L} = \mathbb{Z}\, a_{1,j} + \mathbb{Z}\, a_{2,j}$. Assume M_1 is sufficiently large. Then $w_1 + M_1 a_{1,j}$ is divisible by a large power of p and therefore,

$$\zeta^*(-k, w_j + M_1 a_{1,j}; A_j) = \sum_{\rho} \zeta^*(-k, w_j + M_1 a_{1,j} + \rho; A_j)$$

where ρ runs through a suitable set of representatives for $\mathfrak{L}/p\,\mathfrak{L}$ and $\rho \notin \mathfrak{p}\mathfrak{L}$. At $s = -k$, $\zeta^*(-k, w_j + M_1 a_{1,j} + \rho; A_j)$ is a polynomial in $w_j + M_1 a_{1,j} + \rho$ so we can take the limit $M_1 a_{1,j} \underset{p}{\to} -w_j$ inside. Then we have,

$$\lim_{M_1 a_{1,j} \underset{p}{\to} -w_j} \zeta^*(-k, w_j + M_1 a_{1,j} + \rho; A_j) = \sum_{\rho} \zeta^*(-k, \rho; A_j) = \zeta^*(-k, 0; A_j).$$

□

Let us now set

$$h(s, x; A_j) = \zeta^*(s, x; A_j) - \zeta^*(s, 0; A_j), \quad (4.5)$$

so that $h(-k, x; A_j) = H(-k, x; A_j)$. Then,

Lemma 33. *(Duplication formula) With h as defined in* (4.5)*,*

$$\zeta^*(s, 0; A_j) = \frac{1}{2^s - 4} \sum_{\substack{x_i = 0, \frac{1}{2} \\ x \neq 0}} h(s, (x_1, x_2); A_j) \tag{4.6}$$

where the sum is over all possible combinations $x_i = 0$ or $x_i = \frac{1}{2}$ for $i = 1$, 2, provided $x \neq 0$.

Proof.

$$\begin{aligned} 2^s \zeta^*(s, (0,0); A_j) &= \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p} \nmid m_1 \frac{a_{1,j}}{2} + m_2 \frac{a_{2,j}}{2}}}^{\infty} (m_1 \frac{a_{1,j}}{2} + m_2 \frac{a_{2,j}}{2})^{-s} \\ &= \sum_{x_i = 0,\, \frac{1}{2}} \zeta^*(s, (x_1, x_2); A_j) \end{aligned}$$

where the sum is over all possible combinations $x_i = 0$ or $x_i = \frac{1}{2}$ for $i = 1$, 2. Now subtract $4 \times \zeta^*(s, (0,0); A_j)$ from both sides. □

Corollary 34.

$$\zeta^*(s, x; A_j) = h(s, x; A_j) + \frac{1}{2^s - 4} \sum_{\substack{x_i = 0, \frac{1}{2} \\ x \neq 0}} h(s, (x_1, x_2); A_j).$$

Accordingly, we define the p-adic zeta function as

Definition 35.

$$\zeta_p(s, x; A_j) = H(s, x; A_j) + \frac{1}{2^s - 4} \sum_{\substack{x_i = 0, \frac{1}{2} \\ x \neq 0}} H(s, (x_1, x_2); A_j), \tag{4.7}$$

where $s \in \mathbb{Z}_p$. Note $\zeta_p(s, x; A_j)$ is not defined at $s = 2$.

We immediately have our first main theorem,

Theorem 36. *$\zeta_p(s,x;A_j)$ is the unique p-adic zeta function that interpolates $\zeta^*(-k,x;A_j)$, indeed,*

$$\zeta_p(-k,x;A_j) = \zeta^*(-k,x;A_j)$$

on a dense subset of values $k \in \mathbb{Z}_p$, namely $k = 0,1,2,\ldots,\ k \equiv 0 (mod\, p-1)$.

Proof. By Corollary (32), for $k = 0,1,2,\ldots,\ k \equiv 0 (mod\, p-1)$,

$$\begin{aligned}
\zeta_p(s,x;A_j) &= H(-k,x;A_j) + \frac{1}{2^{-k}-4} \sum_{\substack{x_i=0,\frac{1}{2} \\ x\neq 0}} H(-k,(x_1,x_2);A_j) \\
&= \zeta^*(-k,x;A_j) - \zeta^*(-k,0;A_j) + \frac{1}{2^{-k}-4} \sum_{\substack{x_i=0,\frac{1}{2} \\ x\neq 0}} H(-k,(x_1,x_2);A_j) \\
&= \zeta^*(-k,x;A_j).
\end{aligned}$$

□

For completion we show,

Lemma 37. *The set $K = \{k = 0,-1,-2,\ldots,\ \ k \equiv 0(mod\, p-1)\}$ is dense in $\mathbb{Z}_p$.*

Proof. We need to show for all $x \in \mathbb{Z}_p$ and for all $\epsilon > 0$, there exists a $k \in K$ such that $|x-k|_p < \epsilon$. Let $x = \sum_{n=0}^{\infty} a_n p^n$, $0 \leq a_n < p$. Choose N such that $p^{-N} < \epsilon$. Let $k = -(p^N - 1)\sum_{n=0}^{N} a_n p^n$. Then,

$$|x-k|_p = |x - \sum_{n=0}^{N} a_n p^n + p^N \sum_{n=0}^{N} a_n p^n|_p < p^{-N} < \epsilon.$$

□

4.3.2 Derivative of p-adic $\zeta_p(s,x;A_j)$ function at $s=0$.

We next show how to calculate the derivative of the p-adic zeta function $\zeta_p(s,x;A_j)$ at $s=0$ to a large number of p-adic digits. We first show, for $n \in \mathbb{Z}_p$ and $s \in \mathbb{Z}_p$, that the derivative of n^{-s} at $s=0$ exists. We start from the known result that any $n \in \mathbb{Z}_p^{\times}$ can be written as

$$n = \omega(n)\langle n\rangle$$

where $\langle n\rangle \in 1+p\mathbb{Z}_p$ and $\omega(n)$ is the Teichmüller character associated with n, in the sense that $|\omega(n)-n|_p < 1$. Since $\omega(n)$ is a $(p-1)$-st root of unity, and, for us, $n^{-s} = \lim_{k \underset{p}{\rightarrow} -s} n^k$, with $k \equiv 0 (mod\, p-1)$, then we can start from

$$n^{-s} = \langle n\rangle^{-s}, \ \ \langle n\rangle \in 1+p\mathbb{Z}_p.$$

Lemma 38. *(Stark [65]) The derivative of* n^{-s} *at* $s=0$ *is* $-\log_p(n)$.

Proof. We start from $n^{-s} = \langle n\rangle^{-s}$, $\langle n\rangle \in 1+p\,\mathbb{Z}_p$, so we only need to differentiate n^{-s} for $n = 1+d\,p$, $d \in \mathbb{Z}_p$. In this case we have the definition,

$$\log_p(n) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(dp)^j,$$

which satisfies the usual logarithm rules:

$$\log_p(n\,m) = \log_p(n) + \log_p(m); \ \ \log_p(n^{-1}) = -\log_p(n) \implies \log_p(n) = \frac{1}{p-1}\log_p(n^{p-1}).$$

We previously defined the p-adic function $f_p(s) = n^{-s} = \lim_{t \to -s} n^t$, where t runs through a sequence of integers that approach $-s = (p-1)s'$, $s' \in \mathbb{Z}_p$. Since we want $s=0$, we choose the sequence $t = (p-1)bp^a$, $a \to \infty$. Let $n^{p-1} = 1+cp$, $c \in \mathbb{Z}_p$, so that

$$\log_p(n^{p-1}) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(cp)^j.$$

Noting $p|\frac{p^j}{j!}$ for all p, j,, we obtain,

$$\begin{aligned} n^t &= (1+cp)^{bp^a} \\ &= 1 + \sum_{j=1}^{\infty} bp^a(bp^a-1)\ldots(bp^a-j+1)\frac{(cp)^j}{j!} \\ &\equiv 1 + \sum_{j=1}^{\infty}(bp^a)[(-1)^{j-1}(j-1)!]\frac{(cp)^j}{j!}\ mod(\ p^{2a+1}). \end{aligned}$$

Then,

$$\begin{aligned} \frac{n^t - n^0}{t} &\equiv \frac{1}{p-1}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(cp)^j\ (mod\ p^{a+1}) \\ &\equiv \frac{1}{p-1}\log_p(n^{p-1})(mod\ p^{a+1}) \\ &\equiv \log_p(n)(mod\ p^{a+1}). \end{aligned}$$

We take the limit,

$$\lim_{t\to -s}\frac{n^t - n^0}{t} = \frac{n^{-s}-n^0}{-s} \equiv \log_p(n)(mod\ p^{a+1}),$$

to give,

$$\left|\frac{n^{-s}-n^0}{-s} + \log_p(n)\right|_p \leq p^{-a-1}, \text{ for } |s|_p \leq p^{-a}, \tag{4.8}$$

so that $\frac{d}{ds}(n^{-s})\big|_{s=0} = -log_p(n)$.

$\square$

This proof easily generalizes to the result that the derivative at $s = 0$ of the product of elements in $\mathbb{Z}_p$ raised to the power $-s$ is, as expected, simply minus one times the sum of the p-adic logarithms of each of the elements. We now show we can take the derivative inside the limit,

Lemma 39. *The derivative at $s = 0$ of $f(s) = \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} (n+x)^{-s}$ is given by,*

$$\frac{d}{ds} \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} (n+x)^{-s} \big|_{s=0} = - \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} \log_p(n+x).$$

Proof. We begin with the usual definition of a derivative,

$$\frac{d}{ds} f(s) \big|_{s=0} = \lim_{s \to 0} \frac{f(s) - f(0)}{s}.$$

Now

$$\frac{f(s) - f(0)}{s} = \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} \frac{(n+x)^{-s} - 1}{s},$$

so that,

$$f'(0) = \lim_{s \to 0} \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} \frac{(n+x)^{-s} - 1}{s}.$$

Then, with $N_1 < N$,

$$\begin{aligned}
&\big| f'(0) + \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} \log_p(n+x) \big|_p \\
&\leq \lim_{s \to 0} \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N-1} \big| \frac{(n+x)^{-s} - 1}{s} + log_p(n+x) \big|_p \\
&= \lim_{s \to 0} \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=0}^{N_1-1} \big| \frac{(n+x)^{-s} - 1}{s} + log_p(n+x) \big|_p \\
&+ \lim_{s \to 0} \lim_{N \underset{p}{\rightarrow} -x} \sum_{n=N_1}^{N-1} \big| \frac{(n+x)^{-s} - 1}{s} + log_p(n+x) \big|_p \\
&= S_1 + S_2,
\end{aligned}$$

where the first sum, S_1, p-adically approaches zero as $N \to \infty$ by Lemma (38). The second sum becomes

$$\begin{aligned} S_2 &= \lim_{s\to 0} \lim_{N \xrightarrow[p]{} -x} \sum_{n=N_1}^{N-1} \Big| \frac{(n+x)^{-s}-1}{s} + log_p(n+x) \Big|_p \\ &\leq \lim_{s\to 0} \lim_{N \xrightarrow[p]{} -x} \sum_{n=N_1}^{N-1} p^{-a-1}, \;\; \text{(notation as in (4.8))}, \\ &= \lim_{s\to 0} p^{-a-1} \lim_{N \xrightarrow[p]{} -x} (N - N_1). \end{aligned}$$

The two products in this latter expression both p-adically approach 0 for the given limits. $\square$

We can now find the derivative of our p-adic zeta function at $s = 0$. From the definition of ζ_p we have

$$\begin{aligned} \zeta_p'(0,x;A_j) = H'(0,x;A_j) &- \frac{1}{3}\big[H'(0,(\tfrac{1}{2},0);A_j) + H'(0,(0,\tfrac{1}{2});A_j) + H'(0,(\tfrac{1}{2},\tfrac{1}{2});A_j)\big] \\ &- \frac{log_p 2}{9}\big[H(0,(\tfrac{1}{2},0);A_j) + H(0,(0,\tfrac{1}{2});A_j) + H(0,(\tfrac{1}{2},\tfrac{1}{2});A_j)\big] \end{aligned}$$

First we show how to calculate the $H(0,x;A_j)$ terms.

Lemma 40. *At $s = 0$, we have,*

$$H(0,x;A_j) = \zeta(0,x;A_j) - \zeta(0,0;A_j),$$

Proof. By Corollary (32),

$$H(-k,x;A_j) = \zeta^*(-k,x;A_j) - \zeta^*(-k,0;A_j).$$

We can write $\zeta^*(s,x;A_j)$ as

$$\zeta^*(s,x;A_j) = \zeta(s,x;A_j) - \xi(s,x;A_j)$$

where

$$\xi(s,x;A_j) = \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ \mathfrak{p}|m_1a_{1,j}+m_2a_{2,j}+w_j}}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + w_j)^{-s}.$$

We need to show that the value of $\xi(s,x;A_j)$ at $s=0$ is p-adically zero. By a similar method of proof to that in Corollary (32)

$$\xi(s,x;A_j) = \beta^{-s} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} (m_1a_{1,j} + m_2a_{2,j} + w_j)^{-s},$$

where $\mathfrak{p} = (\beta)$ and $|\beta|_p \leq 1$. The values of this analytically continued function at non-positive integers were found in Corollary (12). The special values at $s=-k$, k a non-negative integer, are given by:

$$\xi(-k,\tilde{w};C) = \beta^k \frac{1}{(k+1)(k+2)} B_{k+2}(w_j;A_j),$$

Since $|\beta|_p \leq 1$, by the von Staudt-Clausen theorem, the p-adic valuation of the right side is less than $p^{-k+\alpha}$ for some fixed integer α. The result follows for $k \underset{p}{\rightarrow} 0$ which implies $k \rightarrow \infty$. □

Remark 41. *First, we note that $\zeta(0,x;A_j)$ and $\zeta(0,0;A_j)$ were calculated in Corollary (7).*

Second, in [66], Stark stated the equation

$$\zeta(-k,x;A_j) - \zeta(-k,0;A_j) = \lim_{T\to\infty} \sum_{m_1=1}^{M_1-1} \sum_{m_2=1}^{M_2-1} L(m+x)^k \tag{4.9}$$

implies

$$\zeta^*(-k,x;A_j) - \zeta^*(-k,0;A_j) = \lim_{T\to\infty} \sum_{m_1=1}^{M_1-1} \sum_{\substack{m_2=1 \\ \mathfrak{p}\nmid L(m+x)}}^{M_2-1} L(m+x)^k + E(k) \tag{4.10}$$

where the error $E(k) \to 0$ *as* $k \underset{p}{\to} 0$*. The reason he gave is that, if* k *is large enough, the terms on the right side of* (4.10) *with a factor of* $\mathfrak{p}$ *have a* p*-adic valuation at most* p^{-k}*, and if* k *is approaching* 0 p*-adically, then* p^{-k} *is much much smaller than the* p*-adic valuation of* k *itself, and so both the limit and derivative at* $s = 0$ *exist and consist of the limit and derivative respectively on the right side at* $s = 0$ *with the terms divisible by* $\mathfrak{p}$ *deleted. He applied the same argument to the corresponding difference between two Hurwitz zeta functions, (the case of one linear form and one summation variable), to replace*

$$\zeta(-k,x;a_{1,1}) - \zeta(-k,M+x;a_{1,1}) = \sum_{m=0}^{M-1}((m+x)a_{1,1})^k$$

with

$$\zeta^*(-k,x;a_{1,1}) - \zeta(-k,0;a_{1,1}) = \lim_{M \underset{p}{\to} -x} \sum_{\substack{m=0 \\ p\nmid(m+x)a_{1,1}}}^{M-1} ((m+x)a_{1,1})^k + E(k),$$

where $E(k) \to 0$ *as* $k \underset{p}{\to} 0$.

We have shown in Lemma (38) that the derivative of n^{-s}, $n \in \mathbb{Z}_p$, $s \in \mathbb{Z}_p$ exists at $s = 0$. This means we can now proceed to find the derivative of $H(s,x;A_j)$ at $s = 0$. For $1 \le i \le 2$, for the summation indices m_i of the multiple sum in formula (4.2) for $H(s,x;A_j)$ we set $m_i = b_i + u_i p$, $0 \le b_i \le p-1$, and define c_i in the range $0 \le c_i \le p-1$ such that $c_i + M_i \equiv b_i \pmod{p}$. Then we define a new set of sequences of positive integers, $U_i,\ 1 \leq i \leq r$, taking the p-adic limits:

$$U_i = \frac{M_i + c_i - b_i}{p} = \bar{U}_i - \sum_{t=1}^{i-1} \frac{u_t a_{t,j}}{a_{i,j}},$$

where,

$$\bar{U}_1 = U_1 \underset{p}{\rightarrow} \frac{\frac{-w_j}{a_{1,j}} + c_1 - b_1}{p} \text{ and } \bar{U}_2 \underset{p}{\rightarrow} \frac{\frac{-1}{a_{2,j}}(w_j + b_1 a_{1,j}) + c_i - b_i}{p}.$$

We use the notation,

$$S_{1,\dots,l} = \sum_{u_1=0}^{U_1-1} \dots \sum_{u_l=0}^{U_l-1} 1.$$

Theorem 42. *(Stark [66]) The derivative of $H(s,x;A_j)$ at $s=0$ may be calculated as*

$$H'(0,x;A_j) = -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p} \nmid L_j(b+x)}}^{p-1} (R_1 + R_2 + R_3) \tag{4.11}$$

where,

$$\begin{aligned}
R_1 &= S_{1,2}\ log_p L_j(b+x), \\
R_2 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{p^k}{L_j(b+x)^k} \frac{1}{(k+1)(k+2)} \left[B_{k+2}(0;A_j) - B_{k+2}(\bar{U}_1 a_{1,j};A_j)\right], \\
R_3 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{p^k}{(L_j(b+x))^k} S_1 \frac{B_{k+1}(\bar{U}_2 a_{2,j};a_{2,j})}{k+1}.
\end{aligned}$$

and, as usual, $L_j(t_1,\dots,t_r) = \sum_{i=1}^{r} a_{i,j} t_i$.

Proof. Assuming the limit $T \to \infty$ so that $M_1 \underset{p}{\rightarrow} -\frac{w_j}{a_{1,j}}$ and $M_2 \underset{p}{\rightarrow} -\frac{w_j + a_{1,j}}{a_{2,j}}$,

$$H'(0,x;A_j)\Big|_{s=0}$$

$$= \frac{d}{ds} \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2-1} L_j(m+x)^{-s}\Big|_{s=0}$$

$$= -\sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ \mathfrak{p}\nmid L_j(m+x)}}^{M_2-1} \log_p L_j(m+x)$$

$$= -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x)}}^{p-1} \sum_{\substack{0\le m_1\le M_1-1 \\ m_1\equiv b_1 (mod\, p) \\ 0\le m_2\le M_2-1 \\ m_2\equiv b_2 (mod\, p)}} \log_p \left(L_j(b+x) + L_j(m-b)\right)$$

$$= -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x)}}^{p-1} \sum_{\substack{0\le m_1\le M_1-1 \\ m_1\equiv b_1 (mod\, p) \\ 0\le m_2\le M_2-1 \\ m_2\equiv b_2 (mod\, p)}} \left(\log_p L_j(b+x) + \log_p(1 + \frac{L_j(m-b)}{L_j(b+x)})\right)$$

$$= -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x))}}^{p-1} \sum_{\substack{0\le m_1\le M_1-1 \\ m_1\equiv b_1 (mod\, p) \\ 0\le m_2\le M_2-1 \\ m_2\equiv b_2 (mod\, p)}} \left(\log_p L_j(b+x) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{L_j(m-b)}{L_j(b+x)}\right)^k\right)$$

Let $m_i = b_i + u_i p \quad$ for $1 \le i \le 2$. Each sum on an m_i becomes a sum on u_i in the range $\;0 \le u_i \le U_i$ where U_i is the smallest integer such that $U_i p + b_i \ge M_i$. Then,

$$H'(0, x; A_j)$$

$$= -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x)}}^{p-1} \sum_{\substack{0\le u_1\le U_1-1 \\ 0\le u_2\le U_2-1}} \left(\log_p L_j(b+x) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{\sum_{i=1}^{2} u_i a_{i,j}}{L_j(b+x)}\right)^k p^k\right)$$

$$= -\sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x)}}^{p-1} \sum_{\substack{0\le u_1\le U_1-1 \\ 0\le u_2\le U_2-1}} \log_p L_j(b+x)$$

$$\quad - \sum_{b_1=0}^{p-1} \sum_{\substack{b_2=0 \\ \mathfrak{p}\nmid L_j(b+x)}}^{p-1} \sum_{\substack{0\le u_1\le U_1-1 \\ 0\le u_2\le U_2-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{\sum_{i=1}^{2} u_i a_{i,j}}{L_j(b+x)}\right)^k p^k$$

Since $\log_p L_j(b+x)$ is independent of the u_i's, the first sum on the right side is the first term in the statement of the theorem.

For the second sum, for each i, define c_i in the range $0 \leq c_i \leq p-1$ such that $M_i + c_i \equiv b_i \pmod p$. Then $U_i p + b_i = M_i + c_i$ or $U_i = \frac{M_i+c_i-b_i}{p}$. Then

$$U_1 = \bar{U}_1 \underset{p}{\rightarrow} \frac{\frac{-w_j}{a_{1,j}} + c_1 - b_1}{p}, \;\; U_2 = \bar{U}_2 - u_1 \frac{a_{1,j}}{a_{2,j}}, \;\text{ where }\; \bar{U}_2 \underset{p}{\rightarrow} \frac{\frac{-1}{a_{2,j}}(w_j + b_1 a_{1,j}) + c_i - b_i}{p}.$$

Now, consider for complex s,

$$\begin{aligned}
&\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}(u_1a_{1,j}+u_2a_{2,j})^{-s}\\
&= \sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{\infty}(u_1a_{1,j}+u_2a_{2,j})^{-s} - \sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{\infty}(u_1a_{1,j}+u_2a_{2,j}+U_2a_{2,j})^{-s}.
\end{aligned}$$

The second sum is $\sum_{u_1=0}^{U_1-1}\zeta(s, u_1a_{1,j}+U_2a_{2,j}; a_{2,j})$. The first sum may also be split as follows

$$\begin{aligned}
&\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{\infty}(u_1a_{1,j}+u_2a_{2,j})^{-s}\\
&= \sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}(u_1a_{1,j}+u_2a_{2,j})^{-s} - \sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}(u_1a_{1,j}+u_2a_{2,j}+U_1a_{1,j})^{-s}\\
&= \zeta(s,(0,0);A_j) - \zeta(s,(U_1,0);A_j).
\end{aligned}$$

At $s=-k$, $k=0,1,2,\ldots$, by Corollaries (11) and (12) we obtain

$$\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}(u_1a_{1,j}+u_2a_{2,j})^{k}$$

$$= \frac{1}{(k+1)(k+2)}\big[B_{k+2}(0;A_j) - B_{k+2}(U_1a_{1,j},0;A_j)\big] + S_1\frac{B_{k+1}(u_1a_{1,j}+U_2a_{2,j};a_{2,j})}{k+1}$$

to which we can apply the limits for the U_i to obtain

$$\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}(u_1a_{1,j}+u_2a_{2,j})^k$$
$$= \frac{1}{(k+1)(k+2)}\big[B_{k+2}(0;A_j) - B_{k+2}(\bar{U}_1a_{1,j};A_j)\big] + S_1\frac{B_{k+1}(\bar{U}_2a_{2,j};a_{2,j})}{k+1}.$$

□

Remark 43. *In the case of zeta functions with one linear form, we can generalize both the definition of $\zeta_p(s,x;A_j)$ and its derivative at $s=0$ to the case of r summation variables. The results are recursive formulas based on p-adic zeta functions with $1,\dots,r$ summation variables. The proofs begin with the definition of the p-adic function*

$$H(s,x;A_j) = \lim_{T\to\infty}\sum_{m_1=0}^{M_1-1}\cdots\sum_{m_r=0}^{M_r-1}L_j(m+x)^{-s},\ \ s\in\mathbb{Z}_p,$$

where $\mathfrak{p}\nmid L_j$ and $T\to\infty$ summarizes the taking of r p-adic limits for M_1 to M_r given by

$$M_k \underset{p}{\to} -\frac{w_j+\sum_{i=1}^{k-1}m_ia_{i,j}}{a_{k,j}}.$$

The process for forming a difference equation analogous to (4.3) works in a reverse manner as required by the taking of the p-adic limits, that is, the first step is to form the expression

$$\sum_{m_r=0}^{M_r-1}L_j(m+x)^{-s} = \sum_{m_r=0}^{\infty}L_j(m+x)^{-s} - \sum_{m_r=M_r}^{\infty}L_j(m+x)^{-s},$$

then take the limit

$$M_r \underset{p}{\to} -\frac{w_j+\sum_{i=1}^{r-1}m_ia_{i,j}}{a_{r,j}},$$

to give a final term equal to a constant times a (Hurwitz) zeta function with one summation variable, and so on. The analogous lemmas and theorems in the remainder of this Section (4.3) *then follow relatively easily. However, the relative ease with which these results can be obtained for general r will not prove to be the case for zeta functions with more than one linear form, which, in view of our goal relating to the Gross Conjecture, are our primary concern.*

4.4 p-adic zeta functions with two linear forms

4.4.1 Definition

We now want to define a p-adic zeta function which solves the interpolation problem for a regularized zeta function defined as follows.

Definition 44. *Let*

$$\zeta^*(s, x; A) = \sum_{m_1=0}^{\infty} \sum_{\substack{m_2=0 \\ p \nmid \prod_{j=1}^{2} L_j(m+x)}}^{\infty} \prod_{j=1}^{2} L_j(m+x)^{-s} \quad Re(s) > 1. \tag{4.12}$$

where $L_j = L_j(m + x) = m_1 a_{1,j} + m_2 a_{2,j} + w_j$ and L_1 is the conjugate of L_2. We will also write $LL' = \prod_{j=1}^{2} L_j(m + x)$. Here $A = (a_{i,j})$ is a 2×2 matrix, where the first and second columns are conjugated over a real quadratic number field F.

We have a similar setup to the case of one linear form: We suppose w_j, $a_{1,j}$ and $a_{2,j}$ are all integers in a quadratic field F and that a fixed prime p splits in F, say $p = \mathfrak{p}\mathfrak{p}'$, where, again, if $\beta \in \mathfrak{p}$ then $|\beta|_p < 1$ and $|\beta'|_p = 1$. We choose two sequences of integers M_1 and M_2 where $M_1 \underset{p}{\to} \alpha$ and $M_2 \underset{p}{\to} \beta$ for some α $\beta \in \mathbb{Z}_p$. Later we will choose α and β to suit our purposes. And, finally, as in the remarks following Definition (21), we use $T \to \infty$ to describe the taking of these two p-adic limits.

We want to define, for $s \in \mathbb{Z}_p$ and A and $L_j = L_j(m+x)$ as above, p-adic functions of the form

$$H(s,x;A) = \lim_{T\to\infty} \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ p\nmid L_1L_2}}^{M_2-1} \prod_{j=1}^{2} L_j(m+x)^{-s}, \tag{4.13}$$

We first note that $(L_1L_2)^{-s}$ is well-defined since $p \nmid L_1L_2$, so that $H(s,w_j;A)$ is well-defined if it converges. Again, that follows from Lemma (26). We can now show,

Lemma 45. *$H(s,x;A)$ exists and is continuous in s, x_1 and x_2 (and in w_j).*

Proof. Put $W = \prod_{j=1}^{2} m_1a_{1,j} + m_2a_{2,j} + w_j$. To show convergence, note that if T increases to T', the group of extra terms added to $\sum_{m_1=0}^{M_1-1}\sum_{\substack{m_2=0\\p\nmid W}}^{M_2-1} W^{-s}$ is

$$\sum_{m_1=0}^{M_1'-1}\sum_{\substack{m_2=0\\p\nmid W}}^{M_2'-1}\prod_{j=1}^{2} W^{-s} - \sum_{m_1=0}^{M_1-1}\sum_{\substack{m_2=0\\p\nmid W}}^{M_2-1}\prod_{j=1}^{2} W^{-s} = \sum_{m_1=0}^{M_1-1}\sum_{\substack{m_2=M_2\\p\nmid W}}^{M_2'-1} W^{-s} + \sum_{m_1=M_1}^{M_1'-1}\sum_{\substack{m_2=0\\p\nmid W}}^{M_2'-1} W^{-s}.$$

We claim the p-adic value of this group approaches zero as $T \to \infty$. Consider,

$$\sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=M_2 \\ p\nmid\prod_{j=1}^{2} m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2'-1} \prod_{j=1}^{2}(w_j + \sum_{i=1}^{2} m_ia_{i,j})^{-s}.$$

Fix a value of m_1 and then consider

$$\sum_{\substack{m_2=M_2 \\ p\nmid\prod_{j=1}^{2} m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2'-1} \prod_{j=1}^{2}(w_j + \sum_{i=1}^{2} m_ia_{i,j})^{-s} = (a_{2,1}a_{2,2})^{-s} \sum_{\substack{m_2=M_2\\p\nmid m_2+y_1}}^{M_2'-1}\prod_{j=1}^{2}(m_2+y_j)^{-s}$$

where $y_j = \frac{m_1a_{1,j}}{a_{2,j}} \in \mathbb{Z}_p$.

Since the number of terms in this sum is $M_2' - M_2 \equiv 0 (mod\, p^T)$, we can apply

Lemma (26). Then

$$\Big| \sum_{\substack{m_2=M_2 \\ p\nmid\prod_{j=1}^{2} m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2'-1} \prod_{j=1}^{2}(w_j+\sum_{i=1}^{2} m_ia_{i,j})^{-s}\Big|_p \to 0$$

as $T \to \infty$. The same argument applies to

$$\sum_{m_1=M_1}^{M_1'-1} \sum_{\substack{m_2=0 \\ p\nmid\prod_{j=1}^{2} m_1a_{1,j}+m_2a_{2,j}+w_j}}^{M_2'-1} \prod_{j=1}^{2}(w_j+\sum_{i=1}^{2} m_ia_{i,j})^{-s}.$$

To show $H(s,x;A)$ is continuous in s, we let $|s_1-s_2|_p < p^{-r}$ and $n_j = m_1a_{1,j} + m_2a_{2,j} + w_j$, $n \in \mathbb{Z}_p$. Then,

$$|H(s_1,x;A)-H(s_2,x;A)|_p = \lim_{T\to\infty} \sum_{m_1=0}^{M_1-1} \sum_{\substack{m_2=0 \\ p\nmid\prod_{j=1}^{2} n_j}}^{M_2-1} \prod_{j=1}^{2} |n_j^{-s_1}-n_j^{-s_2}|_p < p^{-r-1}$$

by Lemma (24).

To show $H(s,x;A)$ is continuous in x_1, put $w = x_1$, $L_j = m_1a_{1,j} + m_2a_{2,j} + x_2a_{2,j}$ and let $|w_1-w_2|_p < p^{-r}$. Then, for some value of m_1 and m_2,

$$|H(s,x;A)-H(s,x;A)|_p \le |\prod_{j=1}^{2}(L_j+w_1a_{1,j})^{-s} - \prod_{j=1}^{2}(L_j+w_2a_{1,j})^{-s}|_p < p^{-r},$$

since for $k=-s$, $k \underset{p}{\to} -s$, the difference has a term in (w_1-w_2). Similarly, $H(s,x;A)$ is continuous in x_2. $\square$

We now proceed to interpolate the complex zeta function

$$\begin{aligned}\zeta^*(s,x;A) &= \sum_{m_1=0}^{\infty}\sum_{\substack{m_2=0\\ p\nmid L_1\,L_2}}^{\infty}(L_1(m+x)L_2(m+x))^{-s}\\ &= \sum_{m_1=0}^{\infty}\sum_{\substack{m_2=0\\ p\nmid L\,L'}}^{\infty}(L\,L')^{-s}\end{aligned}$$

where $L_j(m+x) = m_1\,a_{1,j} + m_2\,a_{2,j} + w_j$ and the 2×2 matrix A is now $A = (a_{i,j})$. We will require the following lemma.

Lemma 46. *The complex zeta function*

$$f(s,a,b) = \sum_{n=0}^{\infty}(n+a)^{-s}(n+b)^{-s},\;\; Re(s) > \frac{1}{2},$$

where a, b are positive real numbers, has a meromorphic continuation to the whole complex plane with a simple pole at $s = \frac{1}{2}$.
The special values at $s=-k$, $k = 0,1,2,\ldots$, $k \equiv 0(mod\,p-1)$, are given by

$$f(-k,a,b) = \frac{(k!)^2}{2}\sum_{\substack{j,l\geq 0\\ j+l=2k+1}}\frac{B_l(1)}{j!\,l!}\sum_{\substack{u,v\geq 0\\ u+v=k}}(-1)^j\binom{j}{u}\binom{l-1}{v}[a^{j-u}b^u + b^{j-u}a^u]. \quad (4.14)$$

Proof. We follow Theorem (14) to first obtain,

$$f(s,a,b) = \Gamma(s)^{-2}\int_0^{\infty}\int_0^{\infty}\frac{e^{(1-a)t_1+(1-b)t_2}}{e^{t_1+t_2}-1}(t_1t_2)^{s-1}dt_1\,dt_2.$$

We set $D_1 = \{t\in\mathbb{R}^2\,|\,0\leq t_2\leq t_1\}$ and $D_2 = \{t\in\mathbb{R}^2\,|\,0\leq t_1\leq t_2\}$ to give

$$f(s,a,b) = \Gamma(s)^{-2}\sum_{k=1}^{2}\int_{D_k}\frac{e^{(1-a)t_1+(1-b)t_2}}{e^{t_1+t_2}-1}(t_1t_2)^{s-1}dt_1\,dt_2 = f_1(s,a,b)+f_2(s,a,b).$$

where, after the change of variables $t = u(y)$, $0 < u$, $0 \leq y_2 \leq 1$, $y_1 = 1$ as in

Theorem (14), we have

$$f_1(s,a,b) = \Gamma(s)^{-2} \int_0^\infty du \int_0^1 \frac{e^{u(1-a)+u(1-b)y_2}}{e^{u(1+y_2)}-1}\, u^{2s-1}\, y_2^{s-1}\, dy_2.$$

Using the usual keyhole integration path, we find at $s = -k$, $k = 0, 1, 2, \ldots$, that $f_1(-k, a, b)$ equals the coefficient of $u^{2k}y^k$ in

$$\frac{k!^2}{2} \sum_{j=0}^{\infty} (-1)^j \frac{u^j(a+by)^j}{j!} \sum_{l=0}^{\infty} \frac{B_l(1)}{l!}[u(1+y)]^{l-1}.$$

Similarly, $f_2(-k, a, b)$ equals the coefficient of $u^{2k}y^k$ in

$$\frac{k!^2}{2} \sum_{j=0}^{\infty} (-1)^j \frac{u^j(b+ay)^j}{j!} \sum_{l=0}^{\infty} \frac{B_l(1)}{l!}[u(1+y)]^{l-1}.$$

Adding the corresponding terms gives (4.14). (Note the sum of the corresponding terms with $l = 0$ is included in this equation.) $\square$

We now choose values of α and β for the p-adic limits $M_1 \underset{p}{\to} \alpha$ and $M_2 \underset{p}{\to} \beta$. We put $\alpha = -x_1$ and

$$-\beta = x_2 + \frac{m_1 + x_1}{2}\left(\frac{a_{1,1}}{a_{2,1}} + \frac{a_{1,2}}{a_{2,2}}\right) = x_2 + \frac{m_1 + x_1}{2} tr\left(\frac{a_{1,1}}{a_{2,1}}\right). \tag{4.15}$$

Following the same procedure as in Theorem (31), for $s \in \mathbb{C}$, $s \neq 1$, M_1 and M_2 positive integers, and $L = L(m + x)$, we put

$$S(s, M_1, M_2; A) = \zeta(s, (x_1, x_2); A) - \zeta(s, (x_1 + M_1, x_2); A) - \sum_{m_1=0}^{M_1-1} \sum_{m_2=0}^{M_2-1} N(L)^{-s}.$$

We also define,

$$\Phi(s,t;A) = 2^{s-1} N_0^{-\frac{s}{2}} \zeta(s,t),$$

where $\zeta(s,t) = \sum_{n=0}^{\infty} (n+x)^{-s}$ is the Hurwitz zeta function and $N_0 = N(\frac{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}}{a_{2,1}a_{2,2}})$. By interchanging the columns of A if necessary, we can assume $N_0 > 0$.

Theorem 47. *At $s = -k$, $k = 0, 1, 2, \ldots$, we have*

$$\lim_{M_1 \underset{p}{\to} \alpha} \lim_{M_2 \underset{p}{\to} \beta} S(-k, M_1, M_2; A) = \Phi(-2k, x_1; A).$$

Proof. By the definition of S, we have

$$S(s, M_1, M_2; A) = N(a_{2,1})^{-s} \sum_{m_1=0}^{M_1-1} f(s, a, b),$$

where $a = x_2 + M_2 + (m_1 + x_1)\frac{a_{1,1}}{a_{2,1}}$ and b is the conjugate of a in F. According to Lemma (46), the special values of $f(s, a, b)$ at $s = -k$ are polynomials in a and b, explicitly given by

$$f(-k, a, b) = \frac{(k!)^2}{2} \sum_{\substack{j,l \geq 0 \\ j+l=2k+1}} \frac{B_l(1)}{j!\, l!} \sum_{\substack{u,v \geq 0 \\ u+v=k}} (-1)^j \binom{j}{u} \binom{l-1}{v} [a^{j-u} b^u + b^{j-u} a^u].$$

We have $B_l = 0$ for all odd l except $B_1(1) = \frac{1}{2}$. Hence, the contribution of all terms with l odd is equal to

$$\frac{(k!)^2}{4(2k)!} \binom{2k}{k} 2(ab)^k = \frac{1}{2}(ab)^k.$$

In all other cases, j is odd since l is even. Therefore,

$$a^{j-u} b^u + b^{j-u} a^u = (ab)^u [a^{j-2u} + b^{j-2u}]$$

where $a^{j-2u}+b^{j-2u}$ is divisible by $a+b$ since j is odd. But in our case,

$$a+b=2(x_2+M_2)+(m_1+x_1)tr(\frac{a_{1,1}}{a_{2,1}}),$$

so that $a+b \underset{p}{\rightarrow} 0$ for $M_2 \underset{p}{\rightarrow} \beta$ by our choice of β. Hence the contribution of all the terms with l even vanishes in the limit $M_2 \underset{p}{\rightarrow} \beta$, which implies $M_2 \rightarrow \infty$. Since,

$$\lim_{M_2 \underset{p}{\rightarrow} \beta} a=\frac{m_1+x_1}{2}(\frac{a_{1,1}}{a_{2,1}}-\frac{a_{1,2}}{a_{2,2}})$$

we obtain

$$\lim_{M_2 \underset{p}{\rightarrow} -\beta} S(-k,M_1,M_2)=2^{-2k-1}N_0^k\sum_{m_1=0}^{M_1-1}(m_1+x_1)^{2k}.$$

But, with $\zeta(s,t)$ the Hurwitz zeta function,

$$\begin{aligned}\sum_{m_1=0}^{M_1-1}(m_1+x_1)^{2k} &= \zeta(-2k,x_1)-\zeta(-2k,x_1+M_1)\\ &= -\frac{1}{2k+1}[B_{2k+1}(x_1)-B_{2k+1}(x_1+M_1)].\end{aligned}$$

Using $B_{2k+1}(0)=0$ again, we see that

$$\begin{aligned}\lim_{M_1 \underset{p}{\rightarrow} \alpha}\lim_{M_2 \underset{p}{\rightarrow} \beta} S(-k,M_1,M_2;A) &= -2^{-2k-1}N_0^k\frac{B_{2k+1}(x_1)}{2k+1}\\ &= \Phi(-2k,x_1;A),\end{aligned}$$

$\square$

Recall that by our definition of the ζ^* notation, $S^*(s,M_1,M_2;A)$ refers to the Dirichlet series defining $S(s,M_1,M_2;A)$ where all terms not relatively prime to p have been removed. We now make the following assumption.

Assumption 48.

$$\lim_{M_1 \underset{p}{\rightarrow} \alpha} \lim_{M_2 \underset{p}{\rightarrow} \beta} S^*(-k, M_1, M_2; A) = \Phi^*(-2k, x_1; A).$$

This assumption is true for $k \underset{p}{\rightarrow} 0$ as discussed in Remark (41).

Stark, [65], has shown that the regularized Hurwitz zeta function, $\zeta^*(-2k, x_1)$ is interpolated by the p-adic Hurwitz zeta function $\zeta_p(s, x_1)$, $s \in \mathbb{Z}_p$. This means that the values $\Phi^*(-2k, x_1; A)$ are p-adically interpolated by

$$\Phi_p(s) = \Phi_p(s, x_1; A) = 2^{s-1} N_0^{-\frac{s}{2}} \zeta_p(s, x_1), \;\; s \in \mathbb{Z}_p.$$

Corollary 49. *Under Assumption* (48)*, we have for* $k \equiv 0 (mod\, p-1)$*,* $k = 0, 1, 2, \ldots$*,*

$$\zeta^*(-k, x; A) - \zeta^*(-k, (0, x_2); A) = \Phi_p(-2k, x_1; A) + H(-k, x; A), \tag{4.16}$$

where $H(s, x; A)$ *is the function of the* p*-adic variable* s *that we defined in* (4.13)*.*

Our next goal is to eliminate the term $\zeta^*(-k, (0, x_2); A)$ in (4.16). To this end we write $\tilde{x} = (x_2, x_1)$ and $\tilde{A} = \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{1,1} & a_{1,2} \end{pmatrix}$, where, as usual, $x = (x_1, x_2)$ and $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$. Then, since

$$\zeta^*(-k, x; A) = \zeta^*(-k, \tilde{x}; \tilde{A}), \tag{4.17}$$

we have,

$$\zeta^*(-k, (0, x_2); A) - \zeta^*(-k, (0, 0); A) = \zeta^*(-k, (x_2, 0); \tilde{A}) - \zeta^*(-k, (0, 0); \tilde{A})$$

$$= \Phi_p(-2k, x_2; \tilde{A}) + H(-k, (x_2, 0); \tilde{A}). \quad (4.18)$$

Adding (4.16) and (4.18), we obtain,

$$\zeta^*(-k, (x_1, x_2); A) - \zeta^*(-k, (0, 0); A) = G(-k, x; A)$$

where,

$$G(-k, x; A) = H(-k, x; A) + H(-k, (x_2, 0); \tilde{A}) + \Phi_p(-2k, x_1; A) + \Phi_p(-2k, x_2; \tilde{A}).$$

Finally, to eliminate the term $\zeta^*(-k, (0, x_2); A)$ we develop a duplication formula, as we did in (4.6), in the form

$$\zeta^*(-k, 0; A) = G_0(-k; A)$$

where

$$G_0(s; A) = \frac{1}{2^{2s} - 4} \sum_{\substack{x_i = 0, \frac{1}{2} \\ x \neq 0}} G(s, (x_1, x_2); A).$$

We are now ready to define the p-adic interpolation of the Shintani cone zeta function $\zeta^*(s, x; A)$.

Definition 50. *For $s \in \mathbb{Z}_p$, we define*

$$\zeta_p(s, x; A) = G(s, x : A) + G_0(s, A).$$

We then have our second main theorem,

Theorem 51. *Under assumption* (48), *$\zeta_p(s, x; A)$ is the unique p-adic zeta function* *that interpolates* $\zeta^*(-k,x;A)$, *indeed,*

$$\zeta_p(-k,x;A)=\zeta^*(-k,x;A)$$

on a dense subset of values $k\in\mathbb{Z}_p$, *namely* $k=0,1,2,\ldots$, $k\equiv 0(mod\, p-1)$.

4.4.2 Derivative of p-adic $\zeta_p(s,x;A)$ at $s=0$.

From the definition of ζ_p we immediately have the following theorem. Note that, following the arguments of Stark [66] as discussed in Remark (41), assumption (48) is not required in the case where $s=0$.

Theorem 52. *The derivative of* $\zeta_p(s,x;A)$ *at* $s=0$ *is given by:*

$$\zeta_p'(0,x;A)=G'(0,x;A)+G_0'(0;A). \tag{4.19}$$

We can calculate the terms on the right side of (4.19) as follows. First,

$$G_0'(0;A)=-\frac{1}{9}\sum_{\substack{x_i=0,\frac{1}{2}\\ x\neq 0}}G'(0,(x_1,x_2);A)+\frac{2}{9}\log_p 2\sum_{\substack{x_i=0,\frac{1}{2}\\ x\neq 0}}G(0,(x_1,x_2);A).$$

We will show how to calculate $G'(0,x;A)$ below. The three terms in the second sum are of the form

$$G(0,x;A)=H(0,x;A)+H(0,(x_2,0);\tilde{A})+\Phi_p(0,x_1;A)+\Phi_p(0,x_2;\tilde{A}).$$

Note that,

$$H(0,x;A)=\zeta(0,x;A)-\zeta(0,0;A)+E(T),$$

where, as discussed in remark (41), $E(T)\to 0$ as $T\to\infty$. We can then evaluate the two remaining terms on the right side using Corollary (18). The term $\Phi_p(0,x_1;A)$ is simply $\frac{1}{2}\,B_1(x_1)$ and $\Phi_p(0,x_2;\tilde{A})$ is $\frac{1}{2}\,B_1(x_2)$.

Let us now consider the calculation of $G'(0,x;A)$. We have,

$$G'(0,x;A) = H'(0,x;A) + H'(0,(x_2,0);\tilde{A}) + \Phi_p'(0,x_1;A) + \Phi_p'(0,x_2;\tilde{A}).$$

The term $\Phi_p'(0,x_1;A)$, ($\Phi_p'(0,x_2;\tilde{A})$ is similar), is given by

$$\Phi_p'(0,x_1;A) = (\frac{1}{2}\log_p 2 - \frac{1}{4}\log_p N_0)\zeta_p(0,x_1) + \frac{1}{2}\zeta_p'(0,x_1).$$

As discussed in Section (2.2), Stark [65] showed $\zeta_p'(0,x) = \log_p \Gamma_p(x)$, so both terms in $\Phi_p'(0,x_1;A)$ can be readily calculated.

It remains to calculate $H'(0,x;A)$. We first need to generalize Theorem (42) to a general statement for $M_1 \underset{p}{\rightarrow} \alpha$ and $M_2 \underset{p}{\rightarrow} \beta$ where $\beta = \beta(m_1)$ and $\alpha,\ \beta \in \mathbb{Z}_p$. Recall that $L_j = m_1 a_{1,j} + m_2 a_{2,j} + w_j$. To deal with the primality condition $\mathfrak{p} \nmid L$ for any prime divisor of p in F, we restrict m_1 and m_2 to residue classes $m_1 \equiv a (mod\, p)$, $m_2 \equiv b (mod\, p)$. Then $\mathfrak{p} \nmid L$ if and only if $\mathfrak{p} \nmid a a_{1,j} + b a_{2,j} + w_j$. So it is enough to consider the subseries

$$H_{a,b}(s) = \lim_{M_1 \underset{p}{\rightarrow} \alpha} \sum_{\substack{m_1=0 \\ m_1 \equiv a (mod\, p)}}^{M_1-1} \lim_{M_2 \underset{p}{\rightarrow} \beta} \sum_{\substack{m_2=0 \\ m_2 \equiv b (mod\, p)}}^{M_2-1} L_j(m+x)^{-s}, \quad s \in \mathbb{Z}_p,$$

where $\beta = \gamma + \delta\, m_1$ for fixed p-adic integers α, γ and δ, and a, b are two integers such that $0 \le a,\ b < p$ and $a a_{1,j} + b a_{2,j} + w_j$ is relatively prime to $\mathfrak{p}$. We write,

$$m_1 = a + up, \quad m_2 = b + vp.$$

Then $0 \le m_2 < M_2$ is equivalent to $0 \le v < V$ where V is the smallest integer such that $Vp + b \ge M_2$. Similarly, $0 \le m_1 < M_1$ is equivalent to $0 \le u < U$ where U is the smallest integer such that $Up + a \ge M_1$. Next we choose integers c, d such that

$$Up + a = M_1 + c, \;\; Vp + b = M_2 + d, \;\; 0 \le c,\, d < p.$$

Then $M_1 \underset{p}{\to} \alpha$, $M_2 \underset{p}{\to} \beta$ means that

$$U \underset{p}{\to} U_0 = \frac{(\alpha + c - a)}{p} \in \mathbb{Z}_p \text{ and } V \underset{p}{\to} \frac{(\gamma + m_1\delta + d - b)}{p} \in \mathbb{Z}_p.$$

We also set,

$$V_0 = \frac{\gamma + a\delta + d - b}{p} \in \mathbb{Z}_p.$$

We now introduce the p-adic generating series

$$G(x; a_{2,j}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k+1)} B_{k+1}(x; a_{2,j}) \big(\frac{p}{aa_{1,j} + ba_{2,j} + w_j}\big)^k,$$

where the generating function for $B_j(x; a)$ is, as usual,

$$\frac{t\, e^{tx}}{e^{at} - 1} = \sum_{j=0}^{\infty} \frac{B_j(x; a)}{j!}\, t^j,$$

and, with A_j given as usual by $(a_{1,j},\, a_{2,j})^t$,

$$F(x; A_j) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)}\, B_{k+2}(x; A_j) \big(\frac{p}{aa_{1,j} + ba_{2,j} + w_j}\big)^k,$$

where the generating function for $B_j(x; A_j)$ is, as usual,

$$\frac{t^2\, e^{tx}}{(e^{a_{1,j}t} - 1)(e^{a_{2,j}t} - 1)} = \sum_{k=0}^{\infty} \frac{B_k(x; A_j)}{k!}\, t^k.$$

We define, for $A_j = (a_{1,j},\, a_{2,j})^t$ and $\tilde{A}_j = (a_{3,j}, a_{2,j})^t$

$$h_{a,b}(w;\alpha,\beta,A_j) = U_0(V_0+(U_0-1)\frac{\delta}{2})\log_p(aa_{1,j}+ba_{2,j}+w_j)-F(0;A_j)+F(U_0a_{1,j};A_j)+T,$$

where

$$T = \begin{cases} F(V_0a_{2,j}+U_0a_{3,j};\tilde{A}_j)-F(V_0a_{2,j};\tilde{A}_j), & \text{if } a_{3,j}=a_{1,j}+\delta a_{2,j}\neq 0 \\ U_0\, G(V_0a_{2,j};a_{2,j}), & \text{if } a_{3,j}=0. \end{cases}$$

We now prove a generalization of Theorem (42) in which Stark [66] considered only the case $a_{3,j} = 0$.

Theorem 53. *With notation as above,*

$$H'_{a,b}(0) = -h_{a,b}(w;\alpha,\beta,A).$$

Proof. After taking the derivative of $H_{a,b}(s)$ term by term and setting $s = 0$, we need to evaluate the p-adic limit of

$$-\sum_{u=0}^{U-1}\sum_{v=0}^{V-1} log_p[(aa_{1,j}+ba_{2,j}+w_j)+(ua_{1,j}+va_{2,j})p,]$$

in the order $M_2 \underset{p}{\rightarrow} \beta$ and then $M_1 \underset{p}{\rightarrow} \alpha$. We expand the p-adic logarithm in the above sum as,

$$\log_p(aa_{1,j}+ba_{2,j}+w_j)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Big(\frac{ua_{1,j}+va_{2,j}}{aa_{1,j}+ba_{2,j}+w_j}\Big)^k p^k,$$

and choose two supplementary integer sequences W and Ω such that $W \underset{p}{\rightarrow} \gamma$ and

$\Omega \underset{p}{\rightarrow} \delta$. This allows us to write

$$M_2 = W + m_1\Omega = W + (a + up)\Omega,$$

and,

$$V = V_1 + u\Omega, \;\; V_1 = \frac{W + a\Omega + d - b}{p}$$

with,

$$V_1 \underset{p}{\rightarrow} V_0 = \frac{\gamma + a\delta + d - b}{p}$$

as $M_2 \underset{p}{\rightarrow} \beta$. The theorem now follows from the following two lemmas.

Lemma 54. *Let* $a_{3,j} = a_{1,j} + \delta a_{2,j}$. *Then,*

$$\lim_{M_2 \underset{p}{\rightarrow} \beta} \sum_{v=0}^{V-1} (ua_{1,j} + va_{2,j})^k = \frac{1}{k+1}\big[B_{k+1}(ua_{3,j} + V_0 a_{2,j}; a_{2,j}) - B_{k+1}(ua_{1,j}; a_{2,j})\big].$$

Proof. We need only to pass to the limit $M_2 \underset{p}{\rightarrow} \beta$ on the right side of the polynomial identity,

$$\sum_{v=0}^{V-1} (ua_{1,j} + va_{2,j})^k = \frac{1}{k+1}\big[B_{k+1}(ua_{1,j} + (V_1 + u\Omega)a_{2,j}; a_{2,j}) - B_{k+1}(ua_{1,j}; a_{2,j})\big],$$

and observe,

$$ua_{1,j} + (V_1 + u\Omega)a_{2,j} = u(a_{1,j} + \Omega a_{2,j}) + V_1 a_{2,j} \underset{p}{\rightarrow} u(a_{1,j} + \delta a_{2,j}) + V_0 a_{2,j} = ua_{3,j} + V_0 a_{2,j}.$$

□

Lemma 55. *If* $a_{3,j} \neq 0$, *then,*

$$\sum_{u=0}^{U-1} B_{k+1}(ua_{3,j} + x; a_{2,j}) = \frac{1}{k+2}[B_{k+2}(x + Ua_{3,j}; \tilde{A}_j) - B_{k+2}(x; \tilde{A}_j)].$$

Proof. By definition,

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{u=0}^{U-1}\frac{B_k(ua_{3,j}+x;a_{2,j})}{k!}\,t^k &= \sum_{u=0}^{U-1}\frac{te^{t(ua_{3,j}+x)}}{e^{ta_{2,j}}-1}\\ &= \frac{1}{t}\,\frac{t^2\,e^{t(Ua_{3,j}+x)}-t^2\,e^{tx}}{(e^{ta_{2,j}}-1)(e^{ta_{3,j}}-1)}.\end{aligned}$$

□

In the excluded case $a_{3,j}=0$, we have,

$$\sum_{u=0}^{U-1}B_{j+1}(x;a_{2,j}) = U\,B_{j+1}(x;a_{2,j}).$$

We are now ready to pass to the limit $M_1 \underset{p}{\to} \alpha$, or, equivalently, $U \underset{p}{\to} U_0$. First, we observe

$$\sum_{u=0}^{U-1}\sum_{v=0}^{V-1}1 = \sum_{u=0}^{U-1}(V_1+u\Omega) = U\,V_1 + \frac{\Omega\,U(U-1)}{2} \underset{p}{\to} U_0(V_0+\frac{\delta(U_0-1)}{2},$$

which yields the coefficient of $\log_p(aa_{1,j}+ba_{2,j}+w_j)$ in the statement of this Theorem (53). Next we pass to the limit $U \underset{p}{\to} U_0$ in Lemma (55) in the case $a_{3,j}\neq 0$ and obtain,

$$\begin{aligned}&\lim_{U\underset{p}{\to}U_0}\sum_{u=0}^{U-1}\lim_{M_2\underset{p}{\to}\beta}\sum_{v=0}^{V-1}(ua_{1,j}+va_{2,j})^k\\ &\qquad= \frac{1}{(k+1)(k+2)}\Big[B_{k+2}(V_0a_{2,j}+U_0a_{,j}3;\tilde{A}_j)-B_{k+2}(V_0a_{2,j};\tilde{A}_j)\\ &\qquad\qquad - B_{k+2}(U_0a_{1,j};A_j)+B_{k+2}(0;A_j)\Big].\end{aligned}$$

In the case $a_{3,j} = 0$, the right side of this equation is replaced by

$$\frac{1}{(k+1)(k+2)}\Big[B_{k+2}(0; A_j) - B_{k+2}(U_0\, a_{1,j}; A_j)\Big] + \frac{1}{k+1} U_0\, B_{k+1}(V_0\, a_{2,j}; a_{2,j}).$$

This finishes the proof of Theorem (53). $\square$

Remark 56. *The complexity of the calculations of the polynomials B_k grows like a polynomial of degree k. Theorem (53) therefore represents a polynomial time algorithm for the calculation of $H'_{a,b}(0)$ to a specified number of p-adic digits.*

Corollary 57. *In the case of one linear form,*

$$H'(0, x; A_j) = -\sum_{a=0}^{p-1} \sum_{\substack{b=0 \\ \mathfrak{p}\nmid(aa_{1,j}+ba_{2,j}+w_j)}}^{p-1} h_{a,b}(w_j; \alpha, \beta, A_j).$$

In the case of two linear forms, where $A = (A_1, A_2)$, we need to change notation and write

$$h_{a,b}(w; \alpha, \beta, A) = h_{a,b}(w_1, \alpha, \beta, A_1) + h_{a,b}(w_2, \alpha, \beta, A_2)$$

to indicate the dependence on the linear form A_j. Then we have,

$$H'(0, x; A) = -\sum_{a=0}^{p-1} \sum_{\substack{b=0 \\ p\nmid N(aa_{1,1}+ba_{2,1}+w_1)}}^{p-1} h_{a,b}(w_2; \alpha, \beta, A_2).$$

This completes the final step in showing how to calculate the derivative of $\zeta_p(s, x; A)$ at $s = 0$ to a specified number of p-dic digits.

Remark 58. *Note that the choice of $\alpha = -x_1 \in \mathbb{Q}$ and*

$$\beta = -(x_2 + \frac{m_1 + x_1}{2} tr(\frac{a_{1,1}}{a_{2,1}} \in \mathbb{Q}$$

for the p-adic limits $M_1 \underset{p}{\to} \alpha$ *and* $M_2 \underset{p}{\to} \beta$ *means we can remove the restriction that* p *splits in the number field* F *from the calculations in the case of two linear forms.*

A Appendix

A.1 Stark's conjecture

In order to prepare for the statement of the Gross-Stark conjecture, which is a p-adic version of the Stark conjecture, we begin by briefly recalling an important special case of the Stark conjecture. We follow the expositions of Stark's Conjecture in Roblot [54], Dummit [19] and Tate [68]. Let F be a number field and E be an abelian extension of F with $G = Gal(E/F)$. Let S be a fixed finite set of places of F containing the infinite places of F and the finite places ramified in E/F. For $\sigma \in G = Gal(E/F)$, define the partial zeta function by the Dirichlet series

$$\zeta_S(\sigma, s) = \sum_{\substack{(\mathfrak{a},S)=1 \\ \mathfrak{a}\subseteq \mathbb{Z}_F \\ \sigma_{\mathfrak{a}}=\sigma}} N(\mathfrak{a})^{-s}$$

where $\mathfrak{a}$ runs through the integral ideals of F not divisible by any prime ideal contained in S and such that the Artin symbol $\sigma_{\mathfrak{a}}$ is equal to σ. We assume there exists an infinite place v which is totally split in E/F and we fix w, a place of E dividing v. We also assume $Card\ S \geq 2$.

Conjecture 59. *(Stark) Let m be the number of roots of unity contained in E. There exists an S-unit $\epsilon \in E$ such that for all $\sigma \in G$,*

$$log|\sigma(\epsilon)|_w = -m\ \zeta'_S(\sigma, 0). \tag{A.1}$$

Furthermore, $E(\sqrt[m]{\epsilon})/F$ is an abelian extension and if $Card\ S \geq 3$ then ϵ, denoted $\epsilon(E/F, w)$, is a (Stark) unit.

Assuming the conjecture is true, the application to the Hilbert's 12-th problem, (how to construct finite abelian extension fields of any number field F using analytic functions depending only on F), is given by the following:

Theorem 60. *(Roblot, [54]) Let F^{Stark} be the subfield of C generated over F by all the units $\epsilon(E/F, w)$ where E/F runs through the finite abelian extensions of F in which v is totally split and w runs through the infinite places of E dividing v. Then, the maximal real abelian extension of F is contained in F^{Stark} or, equivalently, for any finite real abelian extension L/F, there exist Stark units $\epsilon_1, \ldots, \epsilon_r$ such that $L \subset F(\epsilon_1, \ldots, \epsilon_r)$*

In [61] Stark proved this conjecture for $F = \mathbb{Q}$ and in [64] he proved it, using modular forms, for the case where d is a negative integer. The case of d a positive integer remains an open conjecture. Considerable significance is attached to the fact that $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-d})$ are the only number fields with a finite number of units.

A.2 Gross-Stark conjecture

The following exposition of the Gross-Stark conjecture is included here as the motivation for the importance of being able to calculate the derivative of p-adic zeta functions at $s = 0$. The interest is to verify the Gross conjecture in particular cases to a high degree of p-adic accuracy. The p-adic version of Stark's conjecture is due to Gross. We follow Gross [27] and Dasgupta [16]. Let F be a totally real number field and E be a CM field (a totally imaginary quadratic extension of a totally real field) which is an abelian extension of F. For each place $\mathfrak{P}$ of E we let $E_{\mathfrak{P}}$ denote the completion at $\mathfrak{P}$. If $\mathfrak{P}$ is finite we let $N\mathfrak{P}$ denote the cardinality of the residue field of $E_{\mathfrak{P}}$. The restriction of the usual absolute value map $|.|$ to $E_{\mathfrak{P}}^*$ is the normalized local absolute value for which we have the formulas,

$$\| \alpha \|_{\mathfrak{P}} = \begin{cases} N_{H_{\mathfrak{P}}/\mathbb{R}}(\alpha) & \mathfrak{P} \text{ complex} \\ sign(\alpha).\alpha & \mathfrak{P} \text{ real} \\ (N\mathfrak{P})^{-ord_{\mathfrak{P}}(\alpha)} & \mathfrak{P} \text{ finite.} \end{cases}$$

If we restrict the p-adic absolute value $\| \, . \, \|_p$ to the subgroup $E^*_{\mathfrak{P}}$ we obtain a local absolute value with the formulas,

$$\| \alpha \|_{\mathfrak{P},p} = \begin{cases} 1 & \mathfrak{P} \text{ complex} \\ sign(\alpha) & \mathfrak{P} \text{ real} \\ (N\mathfrak{P})^{-ord_{\mathfrak{P}}(\alpha)} & \mathfrak{P} \text{ finite, not dividing } p \\ (N\mathfrak{P})^{-ord_{\mathfrak{P}}(\alpha)} N_{H_{\mathfrak{P}}/\mathbb{Q}_p}(\alpha) & \mathfrak{P} \text{ divides } p. \end{cases}$$

Let $\mathfrak{p}$ be a prime of the totally real number field F lying above the rational prime p of $\mathbb{Q}$. Let E be a finite abelian extension of F such that $\mathfrak{p}$ splits completely in E. Let S be a finite set of primes/places of F which contain all the archimedean primes, the primes lying above p, and all the primes which ramify in E. Note S contains a (finite) place $\mathfrak{p}$ which splits completely in E. Assume $\#S \geq 3$, since this only excludes the case $E = F = \mathbb{Q}$. Write $R = S - \mathfrak{p}$.

For $\sigma \in G$, define the complex partial zeta functions of E/F relative to the sets S and R as:

$$\zeta_S(\sigma, s) = \sum_{\substack{(\mathfrak{a},S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} N\mathfrak{a}^{-s}; \quad \zeta_R(\sigma, s) = \sum_{\substack{(\mathfrak{a},R)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} N\mathfrak{a}^{-s}, \tag{A.2}$$

where the sums are over all integral ideals $\mathfrak{a} \subset \mathbb{Z}_F$ that are relatively prime to the elements of S and R respectively, and whose associated Frobenius element $\sigma_{\mathfrak{a}}$ is equal to σ. These series converge for $Re(s) > 1$ and have meromorphic continuation to $\mathbb{C}$, being regular outside $s = 1$. They are related by the formula

$$\zeta_S(\sigma, s) = (1 - Norm(\mathfrak{p})^{-s})\zeta_R(\sigma, s). \tag{A.3}$$

Hence, $\zeta_S(\sigma, 0) = 0$ and $\zeta'_S(\sigma, 0) = logNorm(\mathfrak{p}).\zeta_R(\sigma, 0)$ for all $\sigma \in G$. The values of each series at non-positive integers are rational and $m\,\zeta_S(\sigma, s)$ is an integer (Deligne and Ribet [18] and Siegel [59]).

Deligne and Ribet [18] and Cassou-Noguès [11] independently proved the existence of a $\mathbb{Q}_p$-valued function $\zeta_{S,\,p}(\sigma, s)$, meromorphic on $\mathbb{Z}_p$ and regular outside $s = 1$, such that

$$\zeta_{S,\,p}(\sigma, k) = \zeta_S(\sigma, k) \tag{A.4}$$

for non-positive integers $k \equiv 0(mod\ d)$ where $d = [F(\mu_{2p}) : F]$. In particular,

$$\zeta_{S,\,p}(\sigma, 0) = \zeta_S(\sigma, 0) = 0. \tag{A.5}$$

Define the group,

$$U_p = \{\epsilon \in E^* : ||\epsilon||_{\mathfrak{P}} = 1 \text{ if } \mathfrak{P} \text{ does not divide } \mathfrak{p}\}$$

Here $\mathfrak{P}$ ranges over all the finite and archimedean places of E. For each divisor $\mathfrak{P}$ of $\mathfrak{p}$ in H, extend the $\mathfrak{P}$-adic valuation $ord_{\mathfrak{P}} : U_p \to \mathbb{Z}$ to the tensor product $\mathbb{Q}U_p = \mathbb{Q} \otimes U_p \to \mathbb{Q}$.

Proposition 61. *(3.8, [27]) Let $\mathfrak{P}$ be a divisor of $\mathfrak{p}$ in E. Then there is a unique* *element* $u = u(\mathfrak{P})$ *in* $\mathbb{Q}U_p = \mathbb{Q} \otimes_{\mathbb{Z}} U_p$ *such that*

$$-\zeta_S'(\sigma, 0) = log||u^\sigma||_{\mathfrak{P}} \ \text{ for all } \sigma \in G. \tag{A.6}$$

and,

$$\zeta_R(\sigma, 0) = ord_{\mathfrak{P}}(u^\sigma) \ \text{ for all } \sigma \in G. \tag{A.7}$$

Since $\mathfrak{p}$ splits completely in E, we have $E \subset E_{\mathfrak{P}} \cong F_{\mathfrak{p}}$.

Gross made the Conjecture for the same element u that:

Conjecture 62. *(2.12, [27]) The element* u *of Equations* (A.6) *and* (A.7) *satisfies*

$$-\zeta_{S,p}'(\sigma, 0) = log_p||u^\sigma||_{\mathfrak{P},p} = -log_p Norm_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u^\sigma) \ \text{ for all } \sigma \in G. \tag{A.8}$$

Gross [27] proved this conjecture in the case $F = \mathbb{Q}$ using formulas developed by Hurwitz and Ferrero-Greenberg [22], the latter of which was in turn derived from the Gross-Koblitz formula proved in [29]. In [41], page 71, Koblitz gave a further proof of the formula and observed (page 45) that "it would be interesting to find an elementary proof". Such a proof was provided by Robert [53] which needs to be read in conjunction with Robert [52].

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