Review of Final Draft Core Standards

R. James Milgram

What follows are my comments on the final draft of the CCSSI Core Mathematics Standards. There are a number of standards including, but not limited to 1-OA(6), 2-OA(2), 2-NBT(5), 3-OA(7), 3-NBT(2), 4-OA(4), 4-OA(6), 4-NF(1), 4-NF(2), 5-OA(3), 8-G(2), 8-G(4), F-LQE(5), G-SRT(4) that are completely unique to this document, and most of them seem problematic to me. I have repeatedly asked for references justifying the insertions of these or similar standards in previous drafts, but references have not been provided. Consequently, to my knowledge, there is no real research base for including any of these standards in the document.

Basic Arithmetic and Arithmetic Operations.

Here are 1-OA(6), 2-OA(2), 3-OA(7), 2-NBT(5), 3-NBT(2), 4-OA(4), and 4-OA(6):

1.OA(6) Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 \ \ 4 = 13 \ \ 3 \ \ 1 = 10 \ \ 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 \ \ 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

2.OA(2) Fluently add and subtract within 20 using mental strategies.

2 By end of Grade 2, know from memory all sums of two one-digit numbers.

2-NBT(5) Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

3-NBT(2) Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

3-OA(7) Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \ \ 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

4-OA(4) . Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations $8 \ \ ? = 48$, $5 = ? \ \ 3$, $6 \ \ 6 = ?$.

4-OA(6) . Understand division as an unknown-factor problem. For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8.

Note that

- most of these standards have some sort of fluency requirement for operations in a range, but no requirement that the algorithm being used is either general or generalizable. Also,

- note the extremely excessive pedagogical constraints in 1-OA(6), 3-OA(7).

- Note that 4-OA(6) is actually a definition, and part of a definition that is given at least one year earlier in virtually all the high achieving countries at that.

  > Specifically, subtraction is defined in the following way: $a - b$ is that number c, if it exists, so that $b + c = a$, while division is defined by $a \div b$ is that number, c, if it exists, so that $b \times c = a$.

  With these understandings, the students in the high achieving countries only have to learn and master two operations, addition and multiplication, since the other two come along for free. Moreover, this is a key piece of the underpinnings for their success. But we are, instead, given 4-OA(6) which is neither fish nor fowl.

As regards fluency, I note that ultimately with

4-NBT(4) Fluently add and subtract multi-digit whole numbers using the standard algorithm.

5-NBT(5) Fluently multiply multi-digit whole numbers using the standard algorithm.

6-NBT(2) Fluently divide multi-digit numbers using the standard algorithm.

expectations are that students will fluently operate with reasonable variants of the standard algorithms. But what will be the effects of the previous fluency requirements, except long-term confusion about key details of what is to be expected? So we can well imagine average and weaker students using some weird mnemonics to handle operations in certain ranges, and trying to combine this with a kind of dim understanding of how the standard algorithms work.

To further add to the confusion surrounding these core standards, note the following entirely reasonable standards that can only be regarded as *competing with the "fluency" standards within the document.*

2-NBT(6) Add up to four two-digit numbers using strategies based on place value and properties of operations.

2-NBT(7) Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

2-NBT(9) Explain why addition and subtraction strategies work, using place value and the properties of operations *This is almost certainly too advanced for second grade, but indicates a viable direction for student exploration in this and later grades.*

3-NBT(3) Multiply one-digit whole numbers by multiples of 10 in the range 1090 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

4-NBT(5) Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

4-NBT(6) Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

5-NBT(6) Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

The seven standards above would have been exemplary if they had not ocurred after the "fluency" standards for unconstrained algorithms that I had objected to at the beginning of this discussion. Within the document itself, there seems to be a minor war going on, and this is not something we should hand over to our teachers.

The above standards illustrate many serious flaws in the Core Standards. Also among these difficulties are that a large number of the arithmetic and operations, as well as the place value standards are one, two or even more years behind the corresponding standards for many if not all the high achieving countries. Consequently, I was not able to certify that the Core Mathematics Standards are benchmarked at the same level as the standards of the high achieving countries in mathematics.

FRACTIONS

Just as there are serious concerns with the coherence of the Core Standards for basic arithmetic and place-value, there are also concerns with the coherence of the Core Standards for fractions, though here the difficulties are somewhat more subtle. Fractions first appear in grade 3 in Core Standards which is somewhat late by international expectations, but not too out of line.

3-NF(1) Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

3-NF(2) Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

3-NF(3) Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b. Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$). Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form $3 = 3/1$; recognize that $6/1 = 6$; locate 4/4 and 1 at the same point of a number line diagram.

d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

Remark. This "visual fraction model" represents all that is wrong in our standard approach to fractions - an approach that has seldom worked. From the glossary, we have

Visual fraction model. A tape diagram, number line diagram, or area model.

In short, what is done is to use three separate and basically unconnected models for fractions to decide if statements are true, false, or ambiguous. In particular, referring back to 3-NF(3), we have the separate notions of *position on the number line* and *size*. These are initially very different concepts when matched to student experience with numbers. Size refers to counting, but when dealing with fractions, counting is not appropriate except in the "partitive" model, which is abandonded very early in the development of this subject in the high achieving countries.

Indeed, what is done in the high achieving countries is to refer fractions entirely back to the number-line as soon as this becomes feasible – usually sometime in second grade or at the beginning of grade 3 – and not refer to size except in-so-far as a number on the number line to the right of another number is said to be larger.

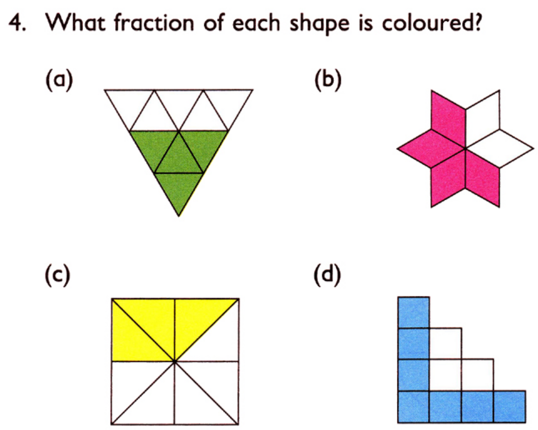
Another point where the handling of fractions is problematic is in fourth grade:

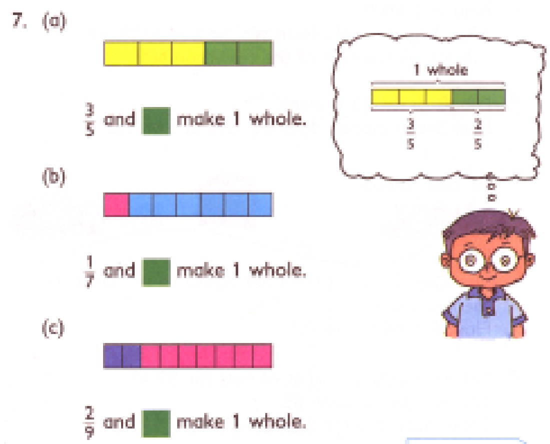
4-FN(1) Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

There are many ways to handle this, but visual fraction models is pretty much the worst. One thing that can be done is to observe that the *point on the number line* associated to $(n \times a)/(n \times b)$ is exactly the same as the point associated to a/b provided $b \neq 0$. So "equivalent fraction" can be taken to mean "fraction representation by the same point on the number line," and, again, in the high achieving countries, this is the approach taken:

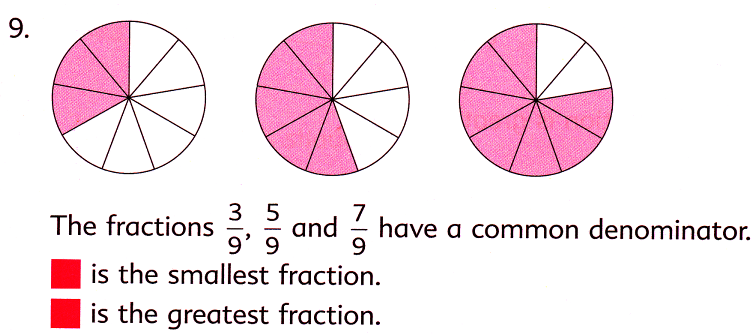
For information, here is the teaching sequence in grades 2 - 4 for fractions in Singapore:

Teaching sequence, Singapore, grades 2 - 4 The initial presentation of fractions in the Singapore programs occurs in the second half of second grade, and is developed using an area model where care is taken to be sure that the regions that decompose a geometric figure are have the same area:

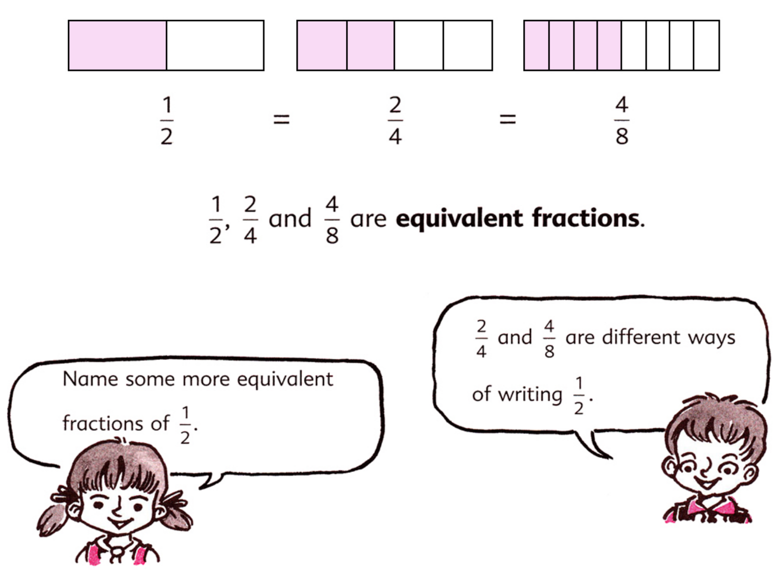




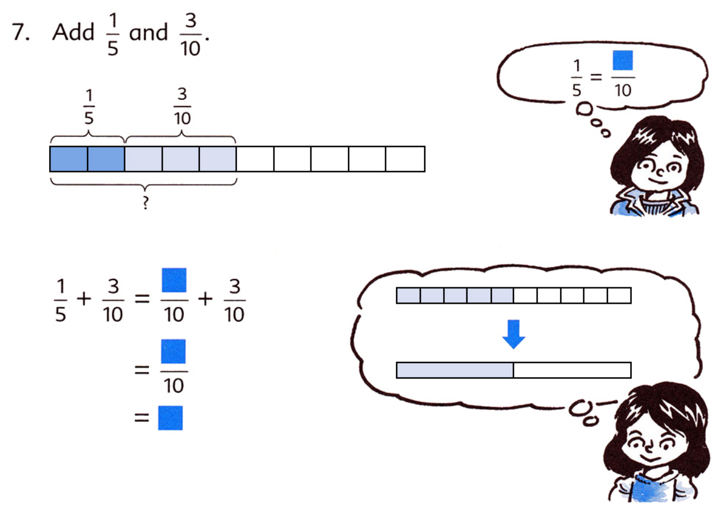
In the second part of the grade three text fractions continue to be developed using an area model, but the level of sophistication as increased significantly:

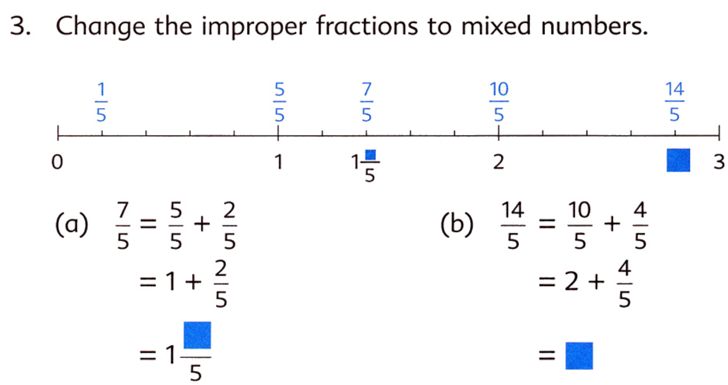


and equaivalent fractions are introduced



In the fourth grade the area model is moved systematically towards seeing fractions on the number line as the basic operations of addition and subtraction of fractions are developed:





Remark: In the Russian texts translated by UCSMP the sequencing is very similar except that representing fractions on the number line is already present in grade 3.

The next problem is with the standard

4-FN(2) Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

The first part of this standard is exemplary, but it is completely distorted by what follows. What does in mean to compare to "a benchmark fraction?" And this is only made worse by the requirement that students "recognize that comparisons are valid only when two fractions refer to the same whole." This is an entirely unappetizing admixture of apples and spoiled oranges.

Geometry

The approach to geometry in Core Standards is very unusual, focusing in eighth grade and beyond on using the Euclidian and extended Euclidean groups to define congruence and similarity as well as develop their key properties. Mathematically, this approach is rigorous, but it would generally be regarded as something that would be done in a college level geometry course for math majors. The exposition at the high school level seems plausible, and may well work. However, to my knowledge, there is no solid research that justifies this approach at the K-12 level currently.

It is also worth noting that a similar approach was taken in Russia about 30 years back, but was quickly rejected. It wasn't that the teachers were not capable of teaching, though this may well be a problem for most middle school and many high school math teachers here. The problem was that it was way too non-standard, and basic geometric facts and theorems had to be handled in entirely new, untested, and ultimately unsuccessful ways.

Here are some details on the issues that arise in geometry.

3-MD(5) Recognize area as an attribute of plane figures and understand concepts of area measurement.

a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

Of course, the basic issue is that most figures in the plane are not decomposed into n unit squares without gaps or overlaps. For example, what of the triangle? 3-MD(5) is a good *beginning* for the discussion of area, but it is not more than this.

In fourth grade we have

4-MD(1) Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a twocolumn table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...

This is the summative standard for a whole sequence of standards that start in the earliest grades but continue through grade 5 or even grade 6. It is far too complex to be listed only in grade 4. But that is exactly what is done in Core Standards. It is as though the authors had a master-list of topics and felt free to sprinkle them wherever there might have been room.

In grade 5 the analogue of 3-MD(5) is presented:

5-MD(3) Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

Partly, I feel that this standard is occuring too early. It takes some time and effort for students to appreciate the complexity of visualizing solid figures through plane sections or possibly nets. Partly, as before, this standard is avoiding the real issues, namely, determining the volumes of figures that can not be decomposed into n cubes without gaps or overlaps, such as triangular prisms or rectangular cones. When we look at this pair of issues together, we can begin to see why I feel so uncomfortable with these standards.

At the same time, look at the geometry standards 5-G(3) and 5-G(4).

5-G(3) Understand that attributes belonging to a category of two dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

5-G(4) Classify two-dimensional figures in a hierarchy based on properties.

Except for parsing the convoluted language of the first, both of these standards are at an astoundingly trivial level for fifth grade. By this time students should be comfortable with the area formula for a triangle, and should be constructing compound two and three dimensional figures as well as determining a number of their properties.

In eighth grade the experimental approach to geometry that I mentioned earlier manifests for the first time. First there is a very superficial development of the properties of some Euclidian transformations in the plane:

8-G(1) Verify experimentally the properties of rotations, reflections, and translations:

a. Lines are taken to lines, and line segments to line segments of the same length.

b. Angles are taken to angles of the same measure.

c. Parallel lines are taken to parallel lines.

Then, based entirely on the relatively weak standard above we are directly given one of the most subtle definitions of congruence we could possibly find.

8-G(2) Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

It is not the first piece of 8-G(2) that disturbs me – though there are a number of key steps that are hidden within it – but the second: given two congruent figures, describe a sequence that exhibits the congruence between them. (By "a sequence" I am presuming the writers meant "a sequence of rotations, reflections and translations.") What is being hidden here is the result that is deep even at the level of a university course in geometry: given two congruent figures, then there exists a Euclidean transformation that takes the first to the second, and a Euclidean transformation that takes the second to the first.

It is at the point above, and even more so with the corresponding similarity standard

8-G(4) Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two dimensional figures, describe a sequence that exhibits the similarity between them

where I feel that we are dealing with an experiment on a national scale. There are even more difficulties with the statement "given two similar two dimensional figures, describe a sequence that exhibits the similarity between them" than was the case with the corresponding statement in 8-G(2).

Before we dare to challenge teachers and students with standards like these, we absolutely have to test the approach in more limited environments, and I find it highly disturbing that H.-H. Wu, the main author of the greometry standards in Core Standards, feels able to make the following statement in a recent article:

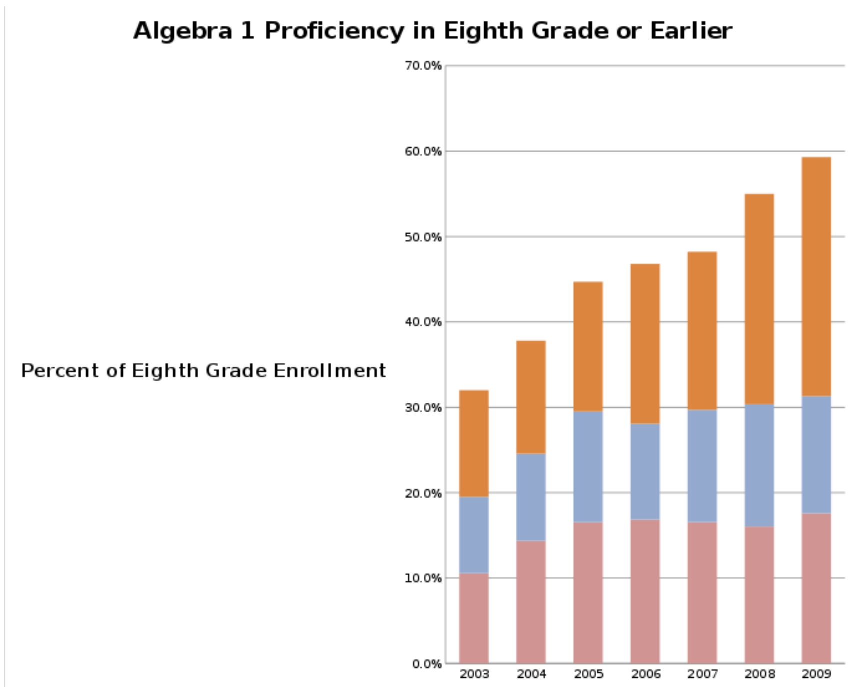
> The mathematical coherence of CCMS also lies at the heart of the discussion of high school geometry. Briefly, the better standards, such as Californias, insist on teaching proofs. This is a good thing, but it does place an unreasonable burden on a high school course on geometry as the only place where any kind of proof can be found in school mathematics. As a result, some of these courses begin with formal proofs based on axioms from the beginning, with no motivation. There is another kind of reaction, however. Giving up entirely on proofs as unlearnable, some courses treat plane geometry as a sequence of hand-on activities that do not mention proofs. In addition, both kinds of courses are disconnected from the teaching of rigid motions (translations, rotations, and reflections) in middle school. What CCMS does is to add the teaching of dilations to rigid motions in grade 8 using hands-on activities, and on this foundation, develops high school geometry by proving all the traditional theorems. For the first time, the school geometry curriculum provides a framework in which all the apparently unrelated pieces of information now begin to form a coherent whole. It holds the promise that learning geometry in K–12 can finally become a reality.

Over the last 12 years Wu and I have collaborated on the California Framework, a number of other states standards, and on a number of nationally influential documents. Normally, Wu is very careful about distinguishing between what one *hopes is true* and what one knows will work, but in this instance I feel he has allowed his innate hope to overwhelm caution.

Eighth Grade Algebra

Another issue with the Core Math Standards is that there are no provisions for eighth grade algebra. This contrasts with the California standards where the expectation is that most students will be ready for Algebra I by eighth grade.

Moreover, as the following graph shows, eighth grade Algebra I is basically working already, with almost 60% of California's students taking the course either in seventh or eighth grade.



It is worth noting that setting standards up so that Algebra I occurs naturally by eighth grade involves a large amount of preparatory material including basic pre-algebra standards and certain key geometry standards, such as understanding that the graph of a linear equation is a straight line. So it is far from sufficient to just list key algebra topics and decree a course that covers them.

As regards the Core Standards this is an issue I, as a member of the Pathways Committee, as well as the Validation Committee, have been struggling with for months. We have been able to rough in courses that are *mostly* based on the High School Core Standards, which will work, but we are far from finished with this project.

Final Remarks

There are also very real strengths in the document. Many of the discussions, among them ratio and rate in grade 6, and proportion in grade 7, are excellent. They are clear and mathematically correct presentations of material that is typically very badly done in most state standards in this country.

Overall, only the very best state mathematics standards, those of California, Massachusetts, Indiana and Minnesota are as strong or stronger than these standards. Most states would be far better off adopting the Core Math Standards than keeping their current standards. However, California, and the other states with top standards would almost certainly be better off keeping their current standards.

In the following pages I detail the comments, organized by grade level, that I had with regard to the final draft of Core Standards. Since many of my objections were not addressed in the two days before the final version was publicly released, the full list may have some use.

Kindergarten

K-CC(1) Count to 100 by ones and by tens. *This is quite a high number for Kindergarten. For example, the CA standard is 30.*

First Grade

1-NBT(1) Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral. *120 is an extremely strange limit. What is the justification for stopping here? To my knowledge, there is no high-achieving country that has 120 as a limit. Also, the CA standard is counting to 100, as are the limits for many of the high achieving countries.*

Compare the first grade California Green dot standard

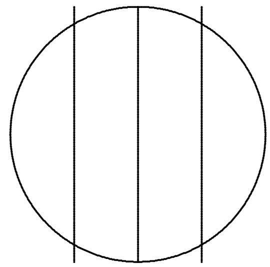
2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

with the much weaker standard in Core Standards:

1-OA(6) Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 - 4 = 13 - 3 - 1 = 10 - 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 - 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13). *This standard misuses "Fluent," I believe. One does not want students to develop fluent command of special tricks for doing arithmetic, as this could very well result in severe difficulties un-learning these methods in later grades where Core Standards asks for at least some degree of proficiency with standard algorithms. Compare 5-NBT(5) Fluently add, subtract, and multiply multi-digit whole numbers using the standard algorithm for each operation.*

1-G(1) The phrase "to possess defining attributes" is almost certainly a misprint. Should probably be *that possess defining attributes.*

1-G(3) Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares. *This seems too early too me to be introducing fractional parts. Also, there is extreme vagueness in the notion of partitioning a circle into 4 equal parts. One often sees* *young students partition the circle as follows:*



Students need a better feel for area than they are likely to have in first grade to handle this standard.

Second Grade

2-OA(1) Use addition and subtraction within 100 to solve one – and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. *Highly unclear what "with unknowns in all positions" means, and the following comments don't seem to help. Overall, this standard contains an excessive amount of jargon. Also, the footnote* {8} *refers to the glossary, but the glossary is not helpful in clarifying these terms.*

2-OA(2) Fluently add and subtract within 20. By end of Grade 2, know from memory all sums of two one-digit numbers. *One year late, when compared with CA and most of the high achieving countries.*

2-OA(3) and 2-OA(4) indirectly are standards for skip counting by 2 and by 5, which is a first grade standard in CA and most high achieving countries.

2-NBT(5) Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. *The same objections hold for this standard as for the first grade standard 1-OA(6). But there are further objections. Place value has been added to the mix here, for the first time, and there should be some indication of what kind of strategies involving place value are expected.*

2-G(2) The phrase "of them" is awkward. Rewrite

2-G(3) How is this different from the firsts grade standard 1-G(3)?

Third Grade

3-OA(1) Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5×7. In view if 3-OA(5), it would seem better to explore both 5×7 and 7×5. Also, note the extra comma.

3-OA(7) Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By end of Grade 3, know from memory all products of one-digit numbers. *As in the earlier grades, "fluently use strategies" is not a standard with research supporting it - especially in view of the fifth (5-NBT(5)) and sixth grade (6-NS(2) standards specifying fluency with the standard algorithms for multiplication and division. The omnibus nature of this standard also tends to minimize the importance of the vital final piece – commit the singe digit multiplication table to memory.*

3-NBT(2) Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. *The same objections apply as applied to similar standards involving "fluently $\cdots$ using strategies, etc." In particular, there is particular danger of students becoming fluent in the use of non-generalizable algorithms that work for a range of numbers – say < 1000 – but do not extend to larger numbers. Also, such algorithms often tend to be very inefficient.*

3-NF(2a) "the endpoint" would be better stated as "the right endpoint."

3-NF(2b) Same comment

3-NF(3a) Recognize and generate simple equivalent fractions (e.g., $1/2 = 2/4$, $4/6 = 2/3$); explain why the fractions are equivalent, e.g., by using a visual fraction model. *At least the term "visual fraction model" is in the glossary. But it is non-standard, and it would be much better to explain it here. Also, how does one justify that the verification of the properties for one of the three fraction models in the visual fraction models, implies that the properties hold for fractions – which are supposed to be numbers, not areas or lengths? This standard and also 3-NF(3b), 3-NF(3c), require more support and preparation.*

3-MD(6) "improvised units." In CA, such measurement units are done with by grade 3, and one uses metric and/or U.S. from grade 3 on.

3-MD(7) It is only after this standard is developed that students have the background for using equal area decompositions as fraction models. Since area decompositions are part of "visual fraction models," this causes problems with the early introduction of fractions.

3-MD(8) Solve real-world and mathematical problems involving perimeters of polygons, such as finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different area or with the same area and different perimeter. *It is almost surely too early for this standard. For example, the last part of this standard first appears as a fourth grade CA standard.*

Very superficial geometry standards. Also in 3-G(1), the term "categories" may have a technical meaning.

Fourth Grade

4-OA(1) Interpret a multiplication equation as a comparison, e.g., interpret $5 \times 7 = 35$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations. *In terms of content, seriously below grade level. Also, an equation is not a comparison (which means "similar to or like".*

4-OA(2) Same misuse of the term "comparison."

4-OA(4) Prime numbers less than 100. There are quite a large number of such primes. In lower grades one almost always restricts the primes being studied to primes < 50.

4-OA(5) Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *What is a pattern? To this point these standards have been reasonably careful about not using undefined terms. But pattern is not in the glossary.*

4-NBT(3) Use place value understanding to round multi-digit whole numbers to any place. *What does this mean? To me, rounding is basically a formal procedure with numbers written in base 10 place value notation. What understanding place value does is to guide students to interpret what the rounding process actually does.*

4-OA(4) Add and subtract multi-digit whole numbers accurately and efficiently using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. *Here there is no fluency required, but at least "accurately and efficiently" is somewhat limiting. However, this is fourth grade and by fourth grade, students in most, if not all, high achieving countries have long since mastered addition and subtraction using standard algorithms, and have understood how and why such algorithms work. Also many have mastered multiplication and division or will complete their mastery of these operations for whole numbers by the end of fourth grade.*

4-OA(5) Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. *This is below the CA fourth grade standards. Here is essentially the same standard though it is somewhat more advanced, but in CA it appears in third grade:*

> *4-3.2 Demonstrate an understanding of, and the ability to use, standard algorithms for multiplying a multi-digit number by a two-digit number and for dividing a multi-digit number by a one-digit number; use relationships between them to simplify computations and to check results.*

4-OA(6) Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. *As indicated this is below expectations in high achieving countries. Moreover, as indicated, it is essentially a third grade CA expectation.*

4-NF(1) Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size; use this principle to recognize and generate equivalent fractions. *In previous version of these standards fractions were defined in terms of special points on the number line. This, at least, gave a single, clear, model for them and a direct way of verifying the properties of fractions. But there are three different models for fractions in the visual fraction models. This has the effect of weakening the coherence of the mathematics students are learning.*

4-NF(2) *The same issues as above occur here.*

4-NF(6) describe a length as 1.62 meters; locate 1.62 on a number line diagram and round 1.62 to 2. *This is a somewhat unnatural rounding. First I would round 1.62 to 1.6 and then 1.6 to 2. So* 2 *is the nearest whole number to 1.62 on the number line, and 1.6 is the closest whole number multiple of* $1/10$ *to* 1.62.

4-MD(5) Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

a) An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $1/360$ of a circle is called a one-degree angle, and can be used to measure angles.

b) An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

It appears that only angles in integer degrees are allowed. What about $\angle A = 36.13°$*?*

4-G(2) Use of the term "category" again. The term is not defined.

Fifth Grade

What is "algebraic thinking?" The term is not in the glossary.

5-OA(1) Interpret grouping symbols in numerical expressions and evaluate expressions with grouping symbols. *What are "grouping symbols?" The term is neither used again, nor is it in the glossary.*

5-OA(3) Generate two numerical patterns using two given rules. Graph pairs of corresponding terms on a coordinate plane, and identify apparent relationships between corresponding terms. *Strange standard. I've never seen one like it and can easily imagine rules that will result in an essentially random difference between terms in the two sequences. For example, Rule 1, each term is 3, rule 2. Take a random number generator R and take the sequence as the successive numbers generated by R.*

5-NBT(3) *Use (5-NBT(2)) to rewrite the expression*

$$347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times \frac{1}{10} + 9 \times \frac{1}{100} + 2 \times \frac{1}{1000}$$

in terms of exponents. Otherwise, why even have 5-NBT(2)?

5-NBT(4) Use place value understanding to round decimals to any place. *See my previous comments on the rounding standards in earlier grades.*

The next three standards, (5-NBT(5), 5-NBT(6) and 5-NBT(7) are low level for fifth grade. This has already been discussed.

5-NF(2) Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. *Compare to 5-NF(1). In 5-NF(1), fractions are treated simply as numbers, which is about right for fifth grade. But in 5-NF(2), it appears that they have gone back to depending on properties of three different sets of models for fractions - a level more appropriate in third or fourth grade. Actually, this is probably not exactly what is meant. I think what is meant is that students should translate word problems involving fractions using tools like Singapore bar diagrams to help them make the translations to mathematical equations. But if this is, in fact, what is meant, then it should have been stated explicitly. As stated, the actual intent of this standard is far from clear.*

In (5-NF(4a)), what is a "story context?" The term is not in the glossary, and only occurs in the fifth and sixth grade standards. There should be a definition.

5-MD(2) Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.* The restriction to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ in fifth grade is far too limiting. In the CA standards, the fourth grade standard (4.1.9) already involves far more complex fractions.

Percents are entirely missing in grade 5. In CA, graphing linear equations already occurs in fifth grade. Also, surface area is measured in fifth grade in CA. The fifth grade geometry standards 5-G(1) - (4) are quite low level when compared to international expectations and the California standards.

6-NS(2) Fluently divide multi-digit numbers using the standard algorithm for each operation. *As has been indicated, it is very late for this standard to appear. In international programs, this typically appears much earlier, no later than fourth grade, and usually earlier than this. In California, the following is a fifth grade standard:*

5-2.2 Demonstrate proficiency with division, including division with positive decimals and long division with multi-digit divisors.

And here is the corresponding fourth grade California standard:

4-3.2 Demonstrate an understanding of, and the ability to use, standard algorithms for multiplying a multi-digit number by a two-digit number and for dividing a multi-digit number by a one-digit number; use relationships between them to simplify computations and to check results.

6-EE(2) Write, read, and evaluate expressions in which letters stand for numbers.

a. Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation "Subtract y from 5" as 5 y.

b. Identify parts of an expression using mathematical language (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression $2 \times (8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.

c. Evaluate expressions by substituting values for their variables, including when using formulas in real-world problems. Perform arithmetic operations (including those involving whole-number exponents) in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$.

This very important standard is mostly fifth grade in California, and somewhat earlier in most high achieving countries. Order of operations should be a separate standard. It is sufficiently important – not as mathematics, since is is purely a sequence of notational conventions – but because these conventions are so universally applied that it is virtually impossible to parse formulas if one is not fluent with them.

6-EE(9) Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *It should be made clearer that it is often impossible to make one variable the dependent variable and the other* *the independent variable. An example like $x^2 + y^2 = 9$ should be included.*

6-G(2) Very similar to 5-MG(5b). Also, here, again, the geometry standards are very superficial. In California, the sixth grade geometry standards include knowing common estimates of π and calculating the circumference and area of circles, recognizing vertical, adjacent, complementary and supplementary angles and their definitions. Also, using their properties and the sum of the angles of a triangle to solve problems involving an unknown angle.

6-SP(5d) Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data was gathered. *This may well be too difficult at the sixth grade level, given the skills and conceptual development assumed to this point.*

Seventh Grade

7-RP(2) Recognize and represent proportional relationships between covarying quantities. *The term "covarying" only appears here. It needs to be defined or this part of the standard should be rephrased.*

7-NS(2d) Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats. *This is a major standard. It should not be the last part of a very long and complex standard, but should be a single standard on its own.*

7-EE(4) Use numbers expressed in the form of a single digit times a whole-number power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *The term "whole number power of 10" should, almost certainly, be "integer power of 10," since this not only includes very large numbers but very small numbers as well.*

7-EE(6a) Solve word problems leading to equations of the form px + q = r and p(x + q) = r, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare the algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *What is meant here when one talks about "algebraic solution?" This term is only used once in the document, and is not in the glossary.*

7-G(5) Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure. *This needs preparation. None of these terms have been developed previously.*

7-SP There is quite a bit of material here all of a sudden.

Eighth Grade

8-NS(1) Understand informally that every number has a decimal expansion; the rational numbers are those with decimal expansions that terminate in 0s or eventually repeat. Know that other numbers are called irrational. *Very similar to 7-NS(2d) above. The only thing that seems to have been added is the definition of irrational. There should probably have been more discussion of what infinite decimals are.*

8-G(2) Understand that a plane figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. *This is a correct definition of congruence, but it is highly non-standard in school mathematics. It might be presented to math majors in a college level course on geometry, but, to my knowledge, there is no research justifying its use in eighth grade. It also hides a very deep geometric theorem, and almost surely will be very difficult for students to figure out and understand.*

8-G(4) Understand that a plane figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar figures, describe a sequence that exhibits the similarity between them. *As stated this is far too similar to 8-G(2). The key difference between them, the use of dilations, needs to be emphasized. Also, the objections and concerns that I had for 8-G(2) hold even more strongly for this standard.*

8-G(5) Use informal arguments to establish facts about the angle sum and exterior angle of triangles, and about the angles created when parallel lines are cut by a transversal. For example, arrange three copies of the same triangle so that the three angles appear to form a line, and give an argument in terms of transversals why this is so. *This is a terrible standard at eighth grade level. It is more appropriate at the fifth or sixth grade level. As has already been pointed out, in the California standards, the sum of the angles of a triangle is a sixth grade standard.*

8-G(6) Explain a proof of the Pythagorean Theorem and its converse. *I don't exactly know what it means to "explain a proof." This standard needs to be clarified.*

8-SP(2) Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. *It is a very bad idea to give the idea that mathematics involves hand-waving. It should be made clear that this is not a mathematics standard, but simply a means to become familiar with curve fitting, and that there are well defined notions of closest line to a bivariate numerical data set.*

8-SP(3) Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height. *The example needs much more discussion. Otherwise, find a different* *example that is more accessible.*

HIGH SCHOOL

The term *explain* is not in the glossary. It is hard to see how such a term can be meaningfully tested with the usual meaning of explain. So there should be clarification about what is meant when this word is used.

A-APR(6) Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. *This is just a rather elaborate way of describing the process of polynomial division with remainder. This should be indicated.*

The two standards A-APR(1) and A-APR(7) are very formal at this level. Some words on why one focuses on these properties might be helpful, but I am concerned that it will be hard to explain this at the high school level.

The two standards A-CED(1) and A-CED(2) seem very low level for high school. They would probably be more appropriate in grade 7 or 8.

F-IF(8a) is repetitive. We have seen completing the square before.

F-BF(2) is somewhat ambiguous. Examples are needed to clarify what is actually intended here.

F-LQE(1(a)) Understand that linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals. *It is quite unclear what "grow by equal factors over equal intervals" actually means. I think this needs to be clarified by the actual formula: "if f is an exponential function, then $f(x+d) = f(x) \times f(d)$ so exponential functions change by equal factors over equal intervals." Also note that grow is probably not the best choice here as $f(d)$ could well be less than 1 in absolute value.*

F-LQE(1c) Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. *Also unclear. It should be revised in a way similar to that indicated for F-LQE(1a).*

F-LQE(3) Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. *Suppose that $f(x) = x^{1+.000003x}$, and $g(x) = x^5 + 11$. Then x will have to be very large before $f(x)$ becomes larger than $g(x)$, and it would be very hard to see this on a graphing calculator or graphing by hand. This standard needs to be clarified and revised.*

F-LQE(5) Interpret the parameters in a linear, quadratic, or exponential function in terms of a context. *This is highly non-standard both in this country and in the high achieving countries. As a consequence, it seems necessary that much more detail be included indicating what kinds of parameters are to be varied and how they should be interpreted.*

F-TF(1) Understand that the radian measure of an angle is the length of the arc on the unit circle subtended by the angle. *What about 4.3π radians?*

F-TF(2) Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. *This is very unclear. The extension of radian measure in terms of the length of arcs on the unit circle to an assignment of angles to all real numbers on the number line involves wrapping the line around the circle – technically, via the map $t \mapsto e^{it}$. But one can illustrate what is happening without getting to the level of understanding e^{it}. In any case, F-TF(2) needs to be revised.*

F-TF(6) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. *I think that the phrase "on the image of this domain" needs to be added at the end.*

G-CO(11) Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. *The use of the term "conversely" is not appropriate here, since the last statement is not the converse of the previous.*

G-SRT(4) Prove theorems about triangles using similarity transformations. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity. *The use of the term "conversely" is not appropriate here, since the last statement is not the converse of the previous.*

G-GPE(3) Derive the equations of ellipses and hyperbolas given two foci for the ellipse, and two directrices of a hyperbola. *The last – two directrices of a hyperbola – is surely wrong, and they don't determine the hyperbola uniquely. I think what may be meant is one directrix and one focus.*