Foundations

of

Plane Geometry

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Prologue

All branches of mathematics are built on a number of postulates or axioms or properties. None of these can be proved, they are simply accepted as facts. It can be argued that the lesser the number of postulates is, the stronger the foundations of that branch of mathematics.

The study of plane geometry or geometry on a plane began with Euclid almost 2,500 years ago. He built his definitions and theorems on the foundations of five postulates. His postulates concerned straight lines, right angles, circles and parallel lines. Four of Euclid's postulates are at the level of definitions or "obvious,"and only his Fifth is a demanding postulate.

Most high school and introductory undergraduate geometry textbooks do not explicitly base plane geometry on Euclid's postulates. They introduce a series of ten other postulates, widening (some would say, weakening) the foundations of plane geometry to fifteen postulates. Accordingly, the purpose of this book is to show how the foundations of plane geometry can be limited to just Euclid's five postulates and that each of the other ten postulates can be proved as theorems.

The first three of the ten concern the relationships between the angles formed by a line cutting across two parallel lines on a plane.

The next three concern pairs of similar triangles, defined as a pair of triangles where the smaller one can be obtained from the other by magnification. For when we put the smaller triangle under a magnifying glass, we see a bigger triangle with three pairs of equal corresponding angles and three pairs of corresponding sides that have been stretched by the same factor so that the three ratios of corresponding pairs of sides are all equal.

We note that the word "equals" is usually replaced in Geometry with the word "congruent." We say two geometric figures are congruent if they are the same shape and size, but we are allowed to flip, slide or turn them to confirm this.

The final four postulates of the ten concern congruent triangles defined as pairs of triangles with corresponding congruent sides and corrresponding congruent angles.

Together with Euclid's five, this gives a total of fifteen unproven postulates on which to base further study of geometry and specifically triangles and several other figures on the plane. But there are a very large number of theorems dealing with triangles and parallel lines, and many of them, and theorems dealing with other plane figures, are proved by invoking one of the triangle congruency or similarity postulates and/or the pairs of parallel lines postulates at some stage. The student is left to depend on a large number of unproven postulates.

This book aims to provide a much firmer foundation on which to build results (theorems) in plane Geometry, accepting only Euclid's five postulates, and particularly his Fifth Postulate, as necessary to prove the other ten postulates are then actually theorems.

We will assume the reader has familiarity with high school algebra including coordinate geometery on the Cartesian Plane but only a minimal background in geometry.

Contents

Chapter 1

Building blocks of Geometry

The building blocks of geometry are points, lines, angles and planes on which we draw figures constructed from points, lines and angles.

1.1 Plane

Definition 1. *Plane*
A plane is a flat surface with length and breadth but no depth.

We therefore say a plane has two dimensions, whereas a box for example has all three dimensions. We say this page is a plane surface but it is actually not a plane since, besides length and breadth, it has thickness or depth. Thus planes are not "things", they are not found in the real world, they are an abstraction from flat surfaces such as this page or the floor of a room.

1.2 Point

Definition 2. *Point*
A point is a location on a plane. It has no dimensions, neither length nor breadth nor depth.

We indicate where a location is on a plane by a "dot" made by using a pen or pencil or keyboard entry. Such a dot has two dimensions (or we could not see it) and is not a point, again a point is an abstraction from a dot on a plane surface used to indicate a location. In Figure 1 the four dots mark four different locations on the plane indicated by the hashed gridlines. To reference them we would of course need an origin and the two usual axes.

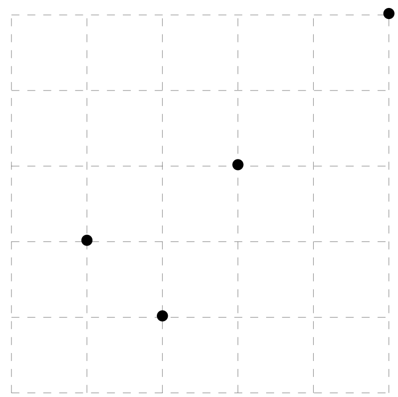


Figure 1: Points on a plane

1.3 Line

The first two of Euclid's five postulates are:

1. *A straight line segment can be drawn joining any two points*

2. *Any straight line segment can be extended indefinitely in a straight line*

This justifies the following procedure and definition. To draw a line on a plane we mark two locations with a dot. Let's call them A and B. We place a straight edge just touching the two points and we use our pencil or other drawing instrument to draw a whole lot more points, guiding our drawing instrument from A to B and onwards and from B to A and onwards, indicating "onwards" with arrows. Thus we have, omitting the gridlines, Figure 2 below.

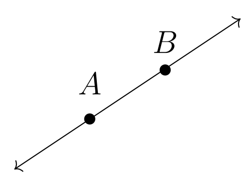


Figure 2: Line through points A and B

Definition 3. *Straight Line*
Given two points A and B, a line containing them is formed by the movement of a point in the constant directions of A to B and then B to A going beyond these two points as far as we like in either direction.

Notation 1. *We write such a line as $\overleftrightarrow{AB}$.*

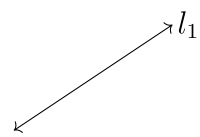


Figure 3: Line

To draw any line in general we can omit the A and B since once we place our straight edge on the plane it borders an infinity of points all linked by the same direction. In general therefore our diagram for a line is as in Figure 3.

Notation 2. *We use l_1, l_2, etc. to name lines in a diagram.*

1.4 Line Segment

Definition 4. *Line segment*
A line segment is a part of a line that joins two points on the line.

Notation 3. *We can use the notation $\overline{AB}$ to mean the line segment joining points A and B but often this makes the diagrams very cluttered so we just use AB.*

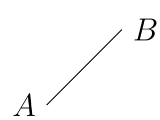


Figure 4: Line Segment $\overline{AB}$ or AB

1.5 Ray

Definition 5. *Ray*
If we begin drawing a line from point A in the direction of any other point and onwards, we have a half-line which we call a ray.

Notation 4.

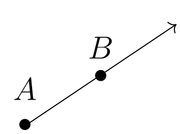


Figure 5: Ray from A containing B

We use the symbol $\overrightarrow{AB}$ to describe a ray beginning at point A and proceeding in the direction of point B and beyond. The diagram for such a ray is shown in Figure 5. Again, a general ray beginning at point A is simply drawn as:

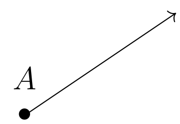


Figure 6: General Ray from Point A

1.6 Angles

Definition 6. *Angle*
We define angles by the following construction on the plane.
We start with a ray in what is called the initial position. The ray can be pointing in any direction, beginning let us suppose, at point O on the plane.

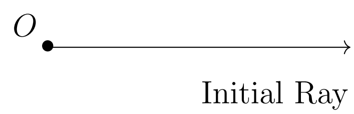


Figure 7: Initial Ray

We imagine a copy of this ray so we that have one overlaying the other. The second one we rotate about the point O to what we call its terminal position, obtaining this diagram.

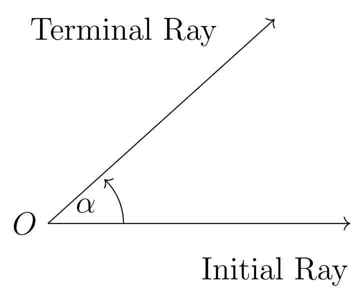


Figure 8: Angle between Initial and Terminal Rays

An angle measure is the measure of how much we have rotated the terminal ray from its starting position "on top of" the initial ray. We use an arc from one ray to the other to indicate the degree of this rotation and we give its measure a name (say the Greek letter alpha, α.)

Notation 5. *As in Figure 8, we will often use the Greek letter α to represent an angle and specifically its measure in degrees. For Figures with more than one angle we will generally use the Greek letters beta, β, gamma, γ, delta, δ. Where there are a large number of angles we will use integers $1, 2, 3, \ldots$.*

Notation 6. *We write* $m\angle ABC = \alpha$ *to mean the measure of angle* $\angle ABC$ *is* α.

Definition 7. *Angle Measurement System*
We define a measurement system for angles by arbitrarily determining that the measure of a complete rotation of one full revolution that puts the terminal ray back on top of the initial ray is $360°$.

Figure 9: Angle of $360°$

Consequently, when the terminal ray is opposite the initial ray in direction we have an angle of measure $180°$ *for the straight line they form.*

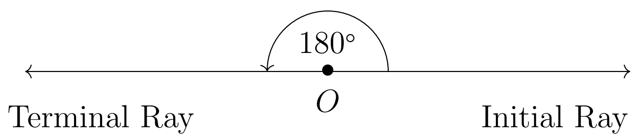


Figure 10: Straight-line angle of $180°$

If we rotate the terminal ray so that it divides the $180°$ *in two, then we have a right angle of measure* $90°$ *which we can label in either of these two ways.*

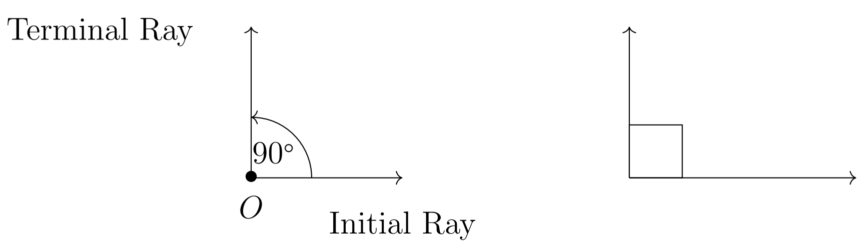


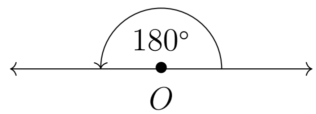
Figure 11: Alternative labels for a right angle

Euclid's fourth postulate is: *All right angles are congruent.*

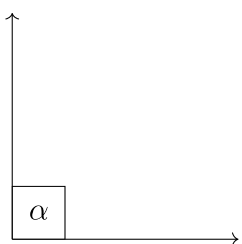
The main types of angles we deal with are summarised as follows.

Notation 7.

1. *Straight Line Angles*

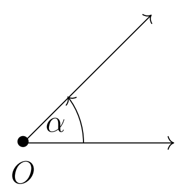


2. *Right Angles*



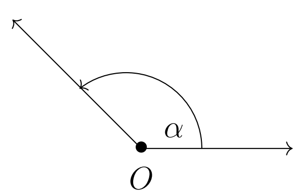
$\alpha = 90°$.

3. *Acute Angles*



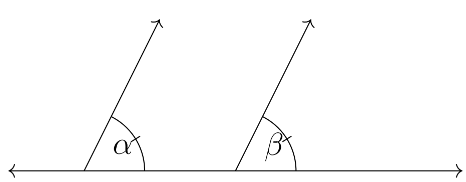
$\alpha < 90°$.

4. *Obtuse angles*



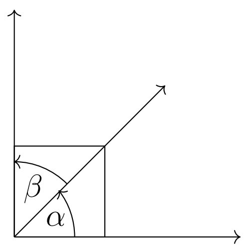
$\alpha > 90°$ *but* $\alpha < 180°$

5. *Congruent Angles (Equal angles)*



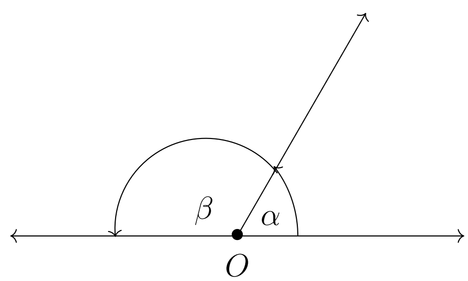
$\alpha = \beta$

6. *Complementary Angles (Two angles that add to a right angle or* $90°$*.)*



$\alpha + \beta = 90°$.

7. *Supplementary Angles (Two angles that add to a straight line angle or* $180°$.)



$\alpha + \beta = 180°$.

Figure 12: Types of Angles

Chapter 2

Transversals and Pairs of Parallel Lines

2.1 Pairs of Intersecting Lines

Two non-parallel lines on the same plane meet or intersect in a common point we call the point of intersection. Our first theorem is:

Theorem 1. *When two lines in the same plane intersect one another, the two pairs of opposite angles are equal in measure.*

Proof. We want to prove, for Figure 13, that $\alpha = \gamma$ and $\beta = \delta$.

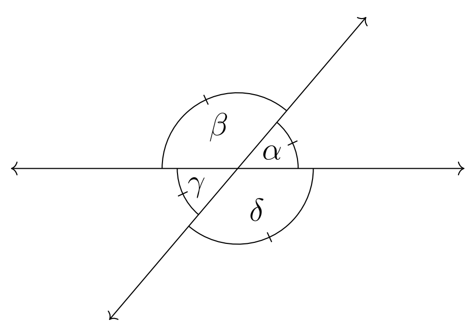


Figure 13: Opposite Angles of a point of intersection

Now α and β add to form the measure of a straight line angle or $180°$.
Hence,

$$\alpha + \beta = 180° \implies \alpha = 180° - \beta.$$

Similarly β and γ add to $180°$ and therefore,

$$\beta + \gamma = 180° \implies \gamma = 180° - \beta$$

Then if $180° - \beta = \gamma$ and also $180° - \beta = \alpha$ then $\alpha = \gamma$.
In the same way $\delta = 180° - \alpha$ so $\beta = \delta$. $\square$

2.2 Euclid's Fifth Postulate

Euclid's fifth postulate is:
If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must invariably intersect each other on that side if extended far enough.
Consider Figure 14.

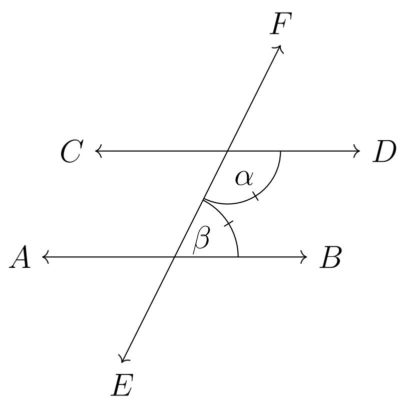


Figure 14: Euclid's Fifth Postulate

With reference to Figure 14, Euclid's Fifth says that if the sum of the measures of the inner angles is $\alpha + \beta < 180°$ then the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ must intersect one another if extended far enough in the $\overrightarrow{AB}$ and $\overrightarrow{CD}$ directions respectively.

Euclid did not mention parallel lines in his postulates but he later defined them by what is now the commonly accepted definition.

Definition 8. *Parallel Lines*
Parallel straight lines are straight lines which, being in the same plane and being extended indefinitely in both directions, do not meet each other in either direction.

Notation 8. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ *means the lines* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$ *are parallel.*

2.3 Three Theorems

If we have a pair of parallel lines cut by a transversal line there are 8 angles formed. For clarity we dispense with the arc symbols on the angles and assign the numbers 1 through 8 as their measures .

Notation 9. *The double arrows on the parallel lines* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$ *indicate they are parallel lines. We may use one or more arrows as circumstances dictate.*

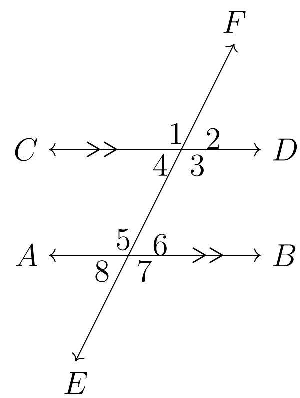


Figure 15: Angles formed by a transversal cutting two parallel lines

Definition 9. *Alternate Interior Angles*
With reference to Figure 15 we define the pair of angles marked 4 and 6 (or the pair 3 and 5) as Alternate Interior Angles.

Definition 10. *Corresponding Angles*
With reference to Figure 15 we define the pair of angles marked 2 and 6 as Corresponding Angles (also the pairs 1,5 and 3,7 and 4,8.)

Definition 11. *Same Side Interior Angles*
With reference to Figure 15 we define the pair of angles marked 3 and 6 (also the pair 4 and 5) as Same Side Interior Angles.

The following is the first key theorem to strengthening the foundations of plane geometry. We explicitly use Euclid's fifth postulate.

Theorem 2. *The alternate interior angles formed by a transversal intersecting two parallel lines are equal (in measure.)*

Proof. We want to prove $\alpha = \beta$ in Figure 16.
We are given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
Assume $\alpha > \beta$.
Then $\alpha + \gamma > \beta + \gamma$
But $\alpha + \gamma = 180°$ (Straight Line Angle)
Hence $\beta + \gamma < 180°$ and they are the measures of the inner angles on one side.
But Euclid's Fifth Postulate states: If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must invariably intersect each other on that side if extended far enough.
Hence $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ will meet. So the assumption $\alpha > \beta$ leads to contradiction to $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and α is not greater than β.

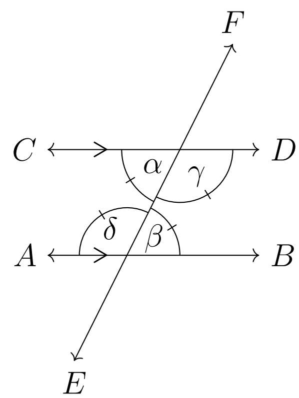


Figure 16: The Four Alternate Interior Angles of the intersections

Next, assume $\beta > \alpha$.
Then $\beta + \delta > \alpha + \delta$
But $\beta + \delta = 180°$. (Straight Line Angle)
Hence $\alpha + \delta < 180°$.
But by Euclid's Fifth Postulate we have another contradiction to $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and it is not true that $\beta > \alpha$.
Hence if it is not true that $\alpha > \beta$ or $\beta > \alpha$ then we must have $\alpha = \beta$.
We have proved the alternate interior angles theorem. $\square$

So, replacing β with α and γ with $180° - \alpha$, and using Theorem 1 on page 8 for the equality of Opposite Angles of intersecting lines, we have Figure 17 below.

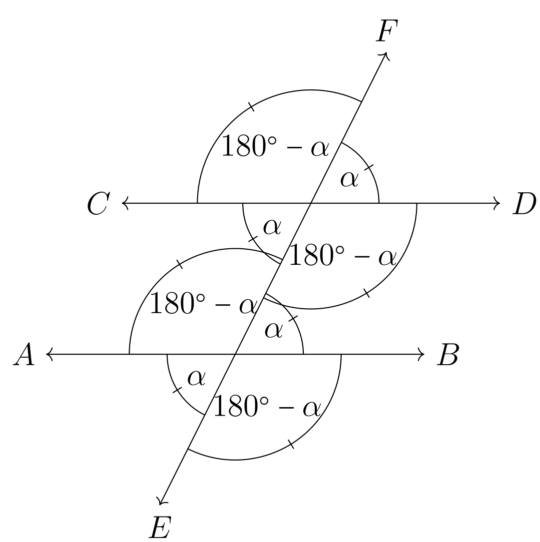


Figure 17: A one variable list of the eight angles

We have built the basis for proving the following two theorems:

Theorem 3. *When a transversal intersects two parallel lines, the pairs of corresponding angles are congruent.*

Proof. Consider Figure 18 below taken from Figure 17. This shows this pair of corresponding angles are congruent and equal in measure. Similarly the other pairs.

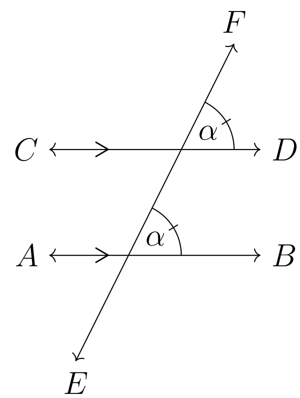


Figure 18: Corresponding Angles of the transversal - parallel lines intersections

□

Theorem 4. *When a transversal intersects two parallel lines, the measures of the two pairs of same side interior angles add to* $180°$.

Proof. Consider the Figure 19 below taken from Figure 17. The measures of the two pairs of same side interior angles add as follows: $180° - \alpha + \alpha = 180°$

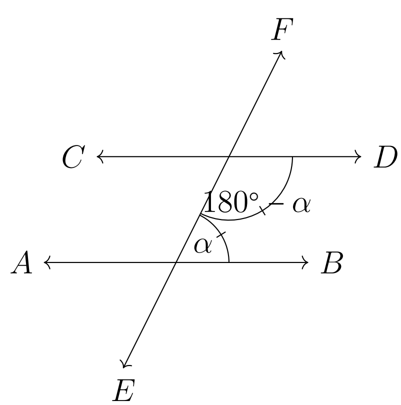


Figure 19: Same Side Interior Angles of the Intersection

□

Chapter 3

Polygons

3.1 Polygons in general

Definition 12. *Polygons*
Polygons are closed figures on a plane formed by a finite number of line segments (at least three) called sides. No two sides are colinear and each side intersects exactly two other sides at their end points.

The sequence of polygons begins thus.

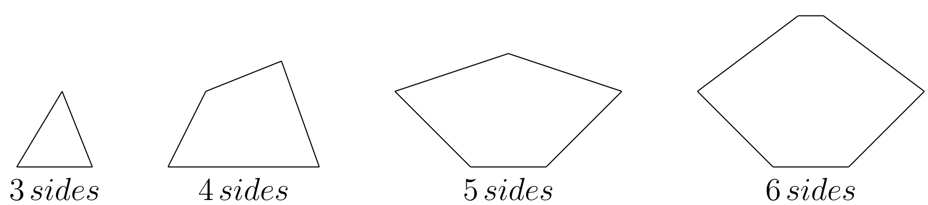


Figure 20: Sequence of polygons

Polygons have the same number of sides, angles and vertices (where two line segments meet). We use specific names for some polygons:

1. Triangle: 3 sides

2. Quadrilateral: 4 sides

3. Pentagon: 5 sides

4. Hexagon: 6 sides

In general we refer to an n-sided polygon as having n sides, angles and vertices.

3.2 Triangles

Definition 13. *Triangle*
A triangle is a plane figure with three line segments forming straight sides, three interior angles and three vertices or points where two line segments meet.

3.2.1 Types of triangles

We name the major categories of triangles as follows.

1. A Scalene triangle has three unequal sides and angles.
2. An Isosceles triangle has two equal sides.
3. An Equilateral triangle has three equal sides.
4. An Acute Angled triangle has three angles all measuring less than 90°.
5. A Right triangle has one angle equal to 90°.
6. An Obtuse angled triangle has one angle measuring greater than 90°.

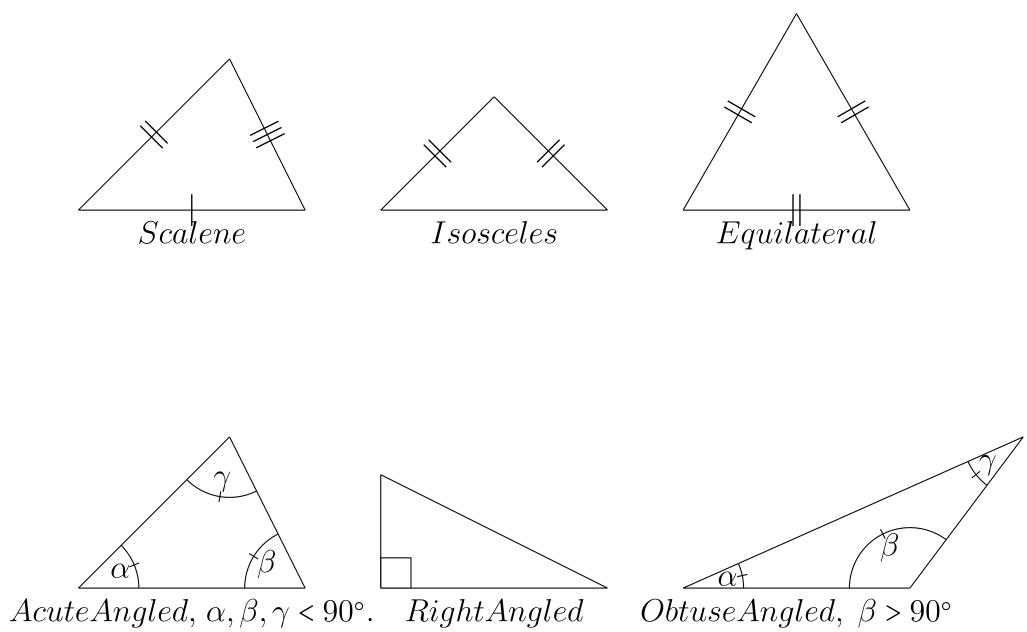


Figure 21: Types of triangles

Notation 10. *We use // and // to indicate two lines have the same length. Also / and ///.*

Notation 11. *A typical triangle is labeled as in Figure 22. The vertices are labeled as A, B and C. The sides opposite the vertices are labeled a, b and c, and mean the length of the line segment on which they are placed. The three angles may be labeled, $\angle A$, $\angle B$ and $\angle C$ or $\angle CAB$, $\angle ABC$ and $\angle ACB$, or they may be labeled with an arc within which is placed a symbol, typically a Greek Letter as is done here with α, β, γ placed by the vertices A, B, C respectively and indicating the measure of an angle.*

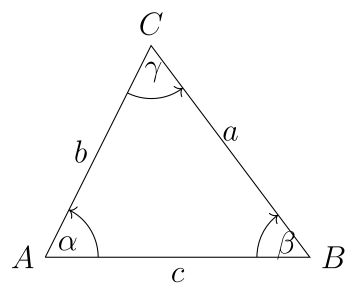


Figure 22: Typical Triangle Labels

3.3 Interior Angles of a Triangle

Definition 14. *Interior Angles of a Triangle*
An interior angle of a triangle is the angle between two sides of the triangle that lies entirely within the triangle.

The interior angles of a triangle are governed by this theorem.

Theorem 5. *The measures of the interior angles of any triangle add to* 180°.

Proof. Consider the triangle ABC below where the three line segments are AB, AC, BC, the three vertices are A, B, C and we mark the three interior angles as α, β, γ in measure and delete the arcs for clarity.

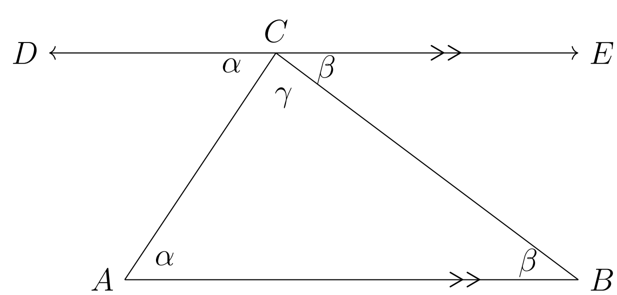


Figure 23: Interior Angles of a Triangle

We choose any vertex, say C, and draw a line $\overleftrightarrow{DE}$ through C parallel to the side AB.

Then by the Alternate Angles Theorem 2 on page 10 we have $m\angle DCA = \alpha$ and $m\angle ECB = \beta$.
But being the angles comprising the straight line angle $\angle DCE$, we have

$$\alpha + \beta + \gamma = 180°$$

Since they are also the interior angles of the triangle, our theorem is proved. □

3.4 Exterior Angles of a Triangle

Definition 15. *Exterior angle of a Triangle*
An exterior angle of a triangle (or any polygon) is formed by extending any side. It is the angle between that extension and the other side that meets it at the vertex. See δ in Figure 24.

Theorem 6. *In measure, an exterior angle of a triangle is equal to the sum of the two interior angles remote from it.*

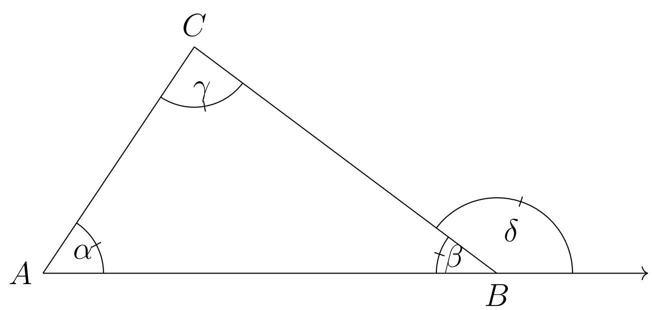


Figure 24: Exterior Angle Theorem for Triangles

Proof. Consider Figure 24.
We want to prove $\delta = \alpha + \gamma$.
By the previous theorem

$$\alpha + \beta + \gamma = 180° \tag{3.4.1}$$

Since they are the measures of a straight line angle,

$$\beta + \delta = 180° \tag{3.4.2}$$

Subtracting (3.2.1)-(3.2.2),

$$\alpha + \gamma - \delta = 0 \implies \delta = \alpha + \gamma$$

□

3.5 Polygons' Angles in general

Theorems 7 and 8 are the only theorems we will prove that are not essential to the argument - the foundations of plane geometry. Why include them? Well, for this reason!

Mathematicians are delighted when they can generalize a theorem to cover an infinite number of cases. We have proved the interior angles of a triangle sum to 180° and from its proof we can prove a general theorem concerning the sum of the interior angles of the general n-sided polygon. Also we have proved a theorem relating to the exterior angles of a triangle and now we can extend that to the sum of the exterior angles of the general n-sided polygon.

Theorem 7. *The sum of the measures of the interior angles of a polygon with N sides is* $(N - 2) \times 180°$

Proof. Note if $N = 3$ then $(N-2)\times 180° = 180°$ as we proved in the preceding theorem. Suppose the N vertices of an N-sided polygon are used as end points of line segments along with some central point (any central point). Figure 25 shows this as dashed lines from the vertices of a 7-sided polygon to such a central point.
All the interior angles of the polygon will comprise the interior angles of N triangles, adding in measure to $N \times 180°$, less the measures of the angles in the circle. But the angles in the circle add to 360°.
Therefore, all interior angles measure $N \times 180° - 360° = (N - 2) \times 180°$

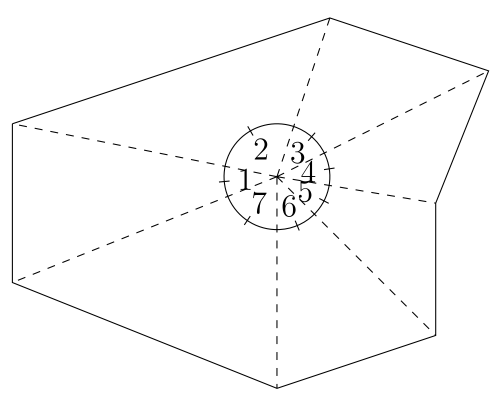


Figure 25: Interior Angles of a Polygon Theorem

□

Theorem 8. *The measures of the exterior angles of all polygons sum to* 180°.

Proof. Figure 26 shows a 5-sided polygon but the proof clearly extends to any N-sided polygon and will be presented thus.

The sum of the measures of the interior angles of an N-sided polygon is $(N-2)\times 180°$ by Theorem 7 on page 17 so,

$$2+4+6+8+10+\cdots=(N-2)\times 180°$$

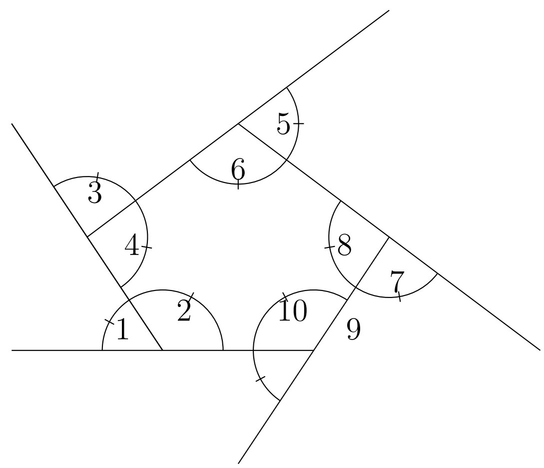


Figure 26: Exterior Angle Theorem for Polygons

In Figure 26, the measure of each of the pairs of angles marked (1,2), (3,4), (5,6), (7,8), (9,10), being supplementary angles, sums to 180°, and in general, for an N-sided polygon there are N such pairs giving a total of $N\times 180°$.

Hence, the sum of the measures of the exterior angles is,

$$\begin{aligned}1+3+5+7+9+\cdots &= N\times 180° - (2+4+6+8+10+\cdots)\\ &= N\times 180° - (N-2)\times 180°\\ &= 2\times 180°\\ &= 360°\end{aligned}$$

□

Note: Theorems 7 and 8 can be strictly proved by using the method of Mathematical Induction.

Chapter 4

Converse Theorems for Transversals and Parallel Lines

The three converse theorems to Theorems 2,3 and 4 all have a similar proof. We delayed proving these converses immediately after the proofs of their respective theorems since we needed the results proved for triangles.

Theorem 9. *If two lines are intersected by a transversal and the same side interior angles are supplementary (add to* $180°$ *in measure) then the two lines are parallel.*

Proof. Suppose two lines, l_1 and l_2, on the same plane are intersected at points A and B on them respectively by a transversal l_3. Suppose further that the two same side interior angles on the right side of the transversal are supplementary, that is we can label them α and $180° - \alpha$ respectively. Let us assume that l_1 and l_2 meet at some point C. We have Figure 27, with the angle arcs deleted for clarity.. Since we have supposed that l_1 and l_2 meet then we have a triangle and $m\angle C = \gamma$ where $\gamma > 0$. This means that the mesures of the three angles of the triangle add to

$$\alpha + (180° - \alpha) + \gamma = 180° + \gamma \tag{4.0.1}$$

But by Theorem 5 on page 15, the sum of the measures of the interior angles of a

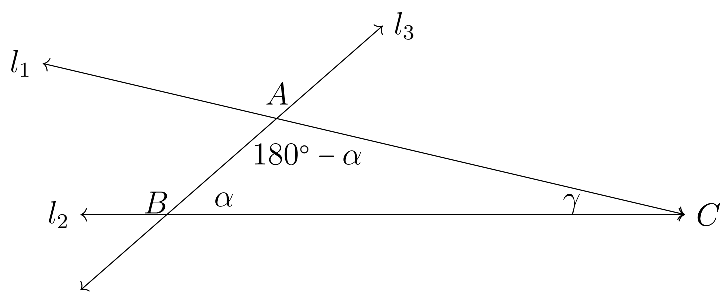


Figure 27: Converse of Same Side Interior Angles Theorem

triangle is 180° so Equation (4.0.1) is impossible. Therefore the two lines l_1 and l_2 do not meet, or, in the words of our definition of parallel lines, they are parallel. ☐

Theorem 10. *If two lines are intersected by a transversal and the alternate interior angles are equal then the two lines are parallel.*

Proof. Two equal alternate interior angles, both labeled α are shown in Figure 28. Then, considering the straight line l_1, $m\angle CAB = 180° - \alpha$.
But then we have two interior same side angles, α and $180° - \alpha$, that add to 180° in measure and by the previous Theorem 9 on page 19, the two lines l_1 and l_2 are parallel.

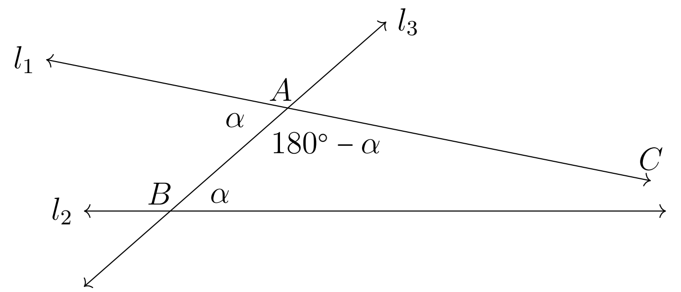


Figure 28: Converse of Alternate Interior Angles Theorem

☐

Theorem 11. *When a transversal line cuts two other lines, if the corresponding angles formed by the transversal and each of the other lines are equal then the two lines are parallel.*

Proof. Two corresponding angles, both labeled α are shown in Figure 29.
Then considering the transversal, l_3, $m\angle CAB = 180° - \alpha$.
But then we have two interior same side angles, α and $180° - \alpha$, that add to 180° in measure and by Theorem 9 on page 19, the two lines l_1 and l_2 are parallel.

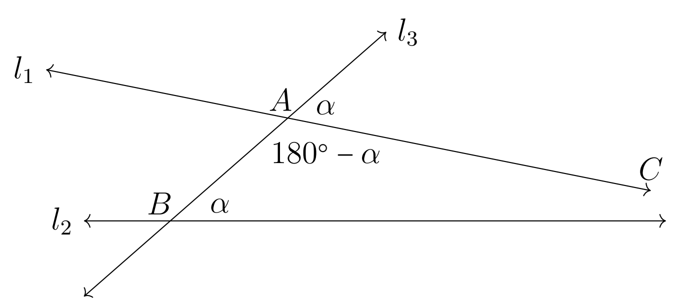


Figure 29: Converse of Corresponding Angles Theorem

☐

Chapter 5

Pythagorean Theorem

5.1 Unit Square and Area

Definition 16. *Unit Square*
A unit square is a quadrilateral with all sides equal to 1 unit and all four angles of measure equal to 90°.

Definition 17. *Area, area of a rectangle, area of a right triangle*
We define the area of a square of side 1 unit to be 1 square unit. See Figure 30.

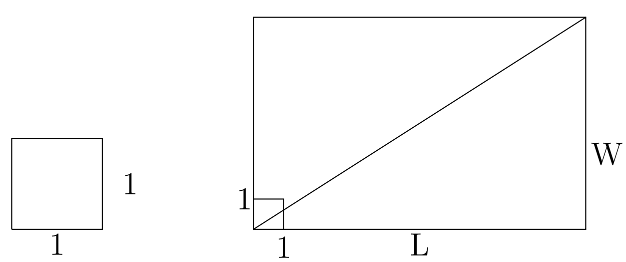


Figure 30: Area

By the area of a rectangle we mean the number of squares of sides 1 unit that it contains. If it is L *units long and* W *units wide then it contains* $L \times W$ *squares of side 1 unit, so its area is,*

$$Area\ of\ rectangle\ =\ L \times W$$

If a diagonal is drawn from one vertex of the square to the opposite vertex, (as in Figure 30) then we have a right triangle of base L *and height* W *which is obviously half the area of the rectangle, so the area of a right triangle is.*

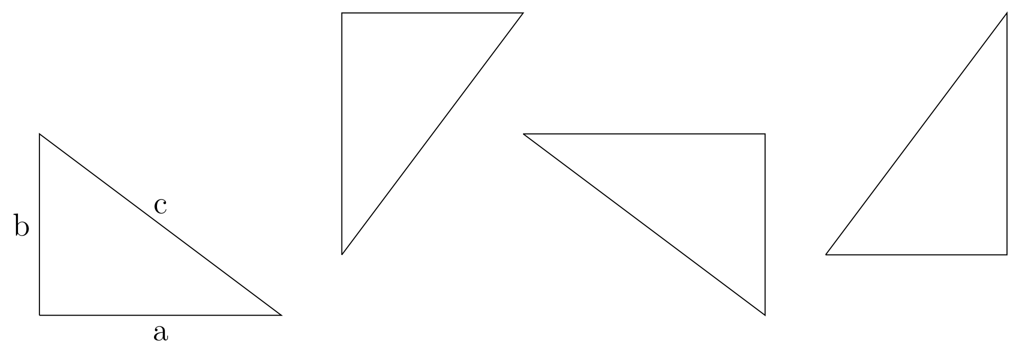
$$Area\ of\ right\ triangle = \frac{1}{2}\ L \times W$$

5.2 Pythagorean Theorem

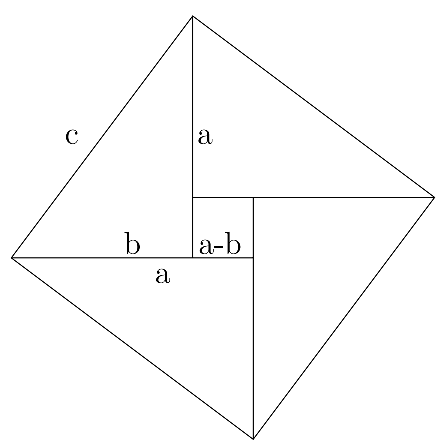
And now for the Greeks' famous theorem.

Theorem 12. *(Pythagorean Theorem)*
In a right triangle, the sum of the squares of the lengths of the two sides forming the right angle is equal to the square of the length of the side opposite the right angle (the hypotenuse).

Proof. We start with four copies of the same right triangle or four congruent triangles. We want to show $a^2 + b^2 = c^2$.



The area of each triangle is $\frac{ab}{2}$ making their combined area $2ab$. We place the four triangles together as shown.



The inner square has sides of length[1] $a - b$ making its area $(a - b)^2$.
The outer square has area c^2. It contains the four triangles and the smaller square, so,

$$(a - b)^2 + 2ab = c^2 \Rightarrow a^2 - 2ab + b^2 + 2ab = c^2 \Rightarrow a^2 + b^2 = c^2$$

□

[1] It makes no difference whether $a > b$ or $a < b$ since $(a - b)^2 = (b - a)^2$.

5.2.1 Applying the Pythagorean Theorem

René Descartes gave us the concept of the coordinate plane with two perpendicular number lines intersecting at their common zero point and called the x and y axis respectively. Locations on this plane can then be identified by an ordered (x, y) pair of numbers called coordinates, the x coordinate of any location being its shortest distance to the x-axis and the y coordinate being its shortest distance to the y-axis. We can easily prove these two shortest distances are the lengths of the line segments that make a right angle with the two axes respectively.

Theorem 13. *The shortest distance from a point to a line is the perpendicular line segment from the point to the line.*

Proof. We want to show the line segment AB is the shortest distance from point A to the line $\overleftrightarrow{DC}$. We draw any other line segment AE from A to the line $\overleftrightarrow{DC}$.

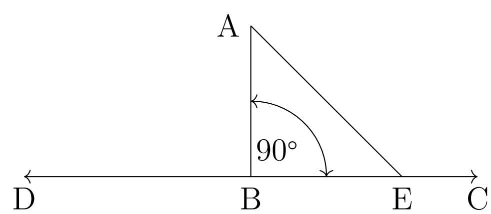


Figure 31: The shortest distance from a point to a line

By the Pythagorean Theorem 12 on page 22,

$$\begin{aligned} &AE^2 = AB^2 + BE^2 \\ &\Longrightarrow AE^2 > AB^2 \\ &\Longrightarrow AE > AB \end{aligned}$$

So the perpendicular line AB is shorter than any other line segment from A to the line $\overleftrightarrow{DC}$. □

5.2.2 Finding distances on the plane

Suppose we have locations P and Q on the Cartesian Plane as shown in Figure 32 below. We want to find their shortest distance apart or the length of the line segment PQ. Using Theorem 13, the coordinates of P are $(3, 2)$ and those of Q are $(-1, -3)$. We form the right triangle PQR where R is the point or location $(3, -3)$. Then $PR = 5$ and $QR = 4$ so by the Pythagorean Theorem, $PQ^2 = 5^2 + 4^2 = 41$ and $PQ = \sqrt{41}$.

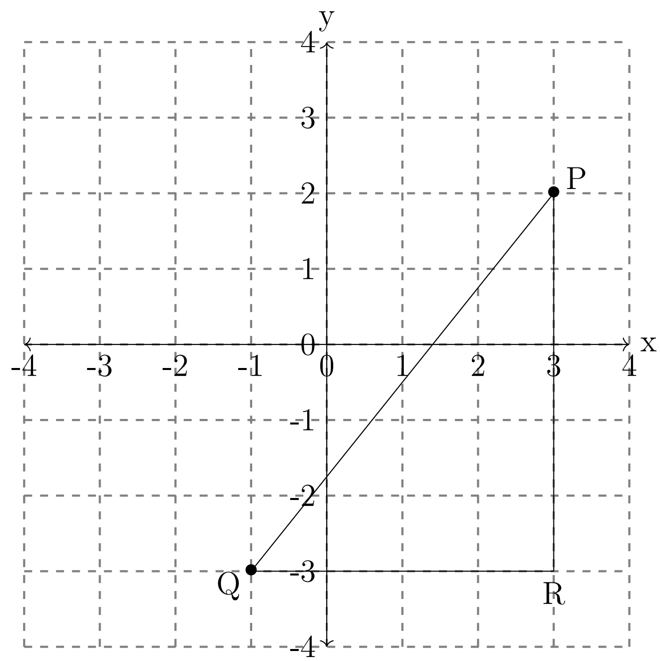


Figure 32: Distances on the Cartesian Plane

5.3 Area of a Triangle

Now that we have a method of calculating distances on a plane we can extend our formula for the area of a right triangle which we saw in Section 5.1 was half the area of a rectangle or $\frac{1}{2}$length × height.

Consider the acute angled triangle $\triangle ABC$ shown in Figure 33. We have drawn the perpendicular CD from C to AB. We call this perpendicular CD the altitude of the triangle or its perpendicular height, and we call AB the base.

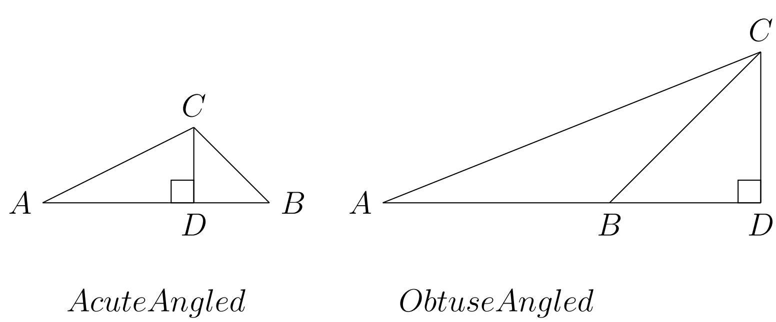


Figure 33: Area of a Triangle

Now,

$$
\begin{aligned}
Area\,of\ \triangle\ ABC &= Area\,of\ \triangle\ ACD + Area\,of\ \triangle\ DCB \\
&= \frac{1}{2}AD \times CD + \frac{1}{2}DB \times CD \\
&= \frac{1}{2}(AD + DB) \times CD \\
&= \frac{1}{2}AB \times CD \\
&= \frac{1}{2}Base \times Perpendicular\,Height.
\end{aligned}
$$

Now consider the obtuse angled triangle in Figure 32. Here,

$$
\begin{aligned}
Area\,of\ \triangle\ ABC &= Area\,of\ \triangle\ ACD - Area\,of\ \triangle\ DCB \\
&= \frac{1}{2}AD \times CD - \frac{1}{2}DB \times CD \\
&= \frac{1}{2}(AD - DB) \times CD \\
&= \frac{1}{2}AB \times CD \\
&= \frac{1}{2}Base \times Perpendicular\,Height.
\end{aligned}
$$

We conclude the area of any triangle is given by $\frac{1}{2}Base \times Perpendicular\,Height.$ or simply,

$$Area\,of\ triangle = \frac{1}{2}bh$$

Chapter 6

An Excursion into Cartesian Geometry

We will need the theorem that the perpendicular distance beween two parallel lines is always the same. If we could use the theorems for similar and congruent triangles, this proof would be easy. But the critically important Thale's theorem depends on the truth of this theorem and in turn the similarity and congruence theorems depend on the truth of Thales' theorem. So we can't use similarity and congruence theorems to prove our result - that would be a circular argument.

6.1 Results from Coordinate Geometry

1. The equation of a line is $y = mx + b$ where, on its graph, m is the slope and b the intercept on the $y-$axis.

2. The equation of a parallel line to $y = mx + b$ is $y = mx + c$ since parallel lines have the same slope but different $y-$intercepts.

3. The slope of a line perpendicular to $y = mx + b$ is $-\dfrac{1}{m}$ and its equation is $y = -\dfrac{1}{m}x + d$ where d is its $y-$intercept on its graph.

4. On the graph shown in Figure 34, the perpendicular line $y = -\dfrac{1}{m}x + d$ will intercept the two lines parallel to it in two points, say A and B. The distance between the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the Cartesian Plane is given by the Pythagorean Theorem:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This information is diagrammed in Figure 34 below.

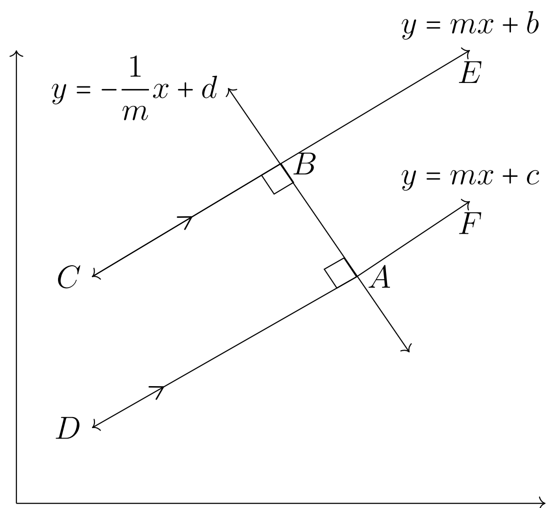


Figure 34: Intersection of perpendicular and parallel lines

6.2 Two theorems

Theorem 14. *The perpendicular distance between two parallel lines is constant.*

Proof. Regarding Figure 34, note if $\angle CBA$ is a right angle then by Theorem 4 on page 12, $\angle DAB$ must also be a right angle.
Figure 34 shows only one perpendicular line cutting the two parallel lines. There are an infinity of such perpendiculars but the only difference in their equations is the value of d or where the perpendicular lines intercept the $y-$axis. If we can show the distance AB is independent of d then we have proved our theorem.
Let us find the coordinates of the points A and B and apply the distance formula to find the length of AB.
To find the coordinates (x_1, y_1) of A we solve as follows:

$$y = mx + c \; and \; y = -\frac{1}{m}x + d \; gives:$$

$$mx + \frac{1}{m}x = d - c$$

$$\implies x_1 = \left(\frac{d-c}{m + \frac{1}{m}} \right)$$

$$Then,\ y = mx + c \implies y_1 = m\left(\frac{d-c}{m+\dfrac{1}{m}}\right) + c$$

Similarly for the coordinates (x_2, y_2) of B we have:

$$y = mx + b\ and\ y = -\frac{1}{m}x + d\ gives:$$
$$mx + \frac{1}{m}x = d - b$$
$$\implies x_2 = \left(\frac{d-b}{m+\dfrac{1}{m}}\right)$$
$$Then,\ \ y = mx + b \implies y_2 = m\left(\frac{d-b}{m+\dfrac{1}{m}}\right) + b$$

Then,

$$x_2 - x_1 = \frac{d-b-d+c}{m+\dfrac{1}{m}} = \frac{c-b}{m+\dfrac{1}{m}}$$
$$y_2 - y_1 = m\left(\frac{d-b-d+c}{m+\dfrac{1}{m}}\right) + b - c = m\left(\frac{c-b}{m+\dfrac{1}{m}}\right) + b - c$$

Accordingly, the distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is independent of d and we have proved our theorem. □

Another result we will need later concerns two parallel lines acting as transversals across another pair of parallel lines. If we let the slope of the transversal parallel lines be m_2 and the slope of the other pair of parallel lines be m_1 then Figure 35 shows this setup. A quadrilateral with opposite sides parallel is called a parallelogram. We want to prove the opposite sides of a parallelogram are equal in length. Usually this is proved using congruent triangles formed by joining a pair of opposite vertices, but we want to use our result to prove theorems about congruent triangles, so again we must avoid a circular argument.

Theorem 15. *The opposite sides of a parallelogram are equal in length.*

Proof. Figure 35 shows a parallelogram $ABCD$ formed by a pair of different parallel transversal lines intersecting another pair of different parallel lines not parallel to the first pair. The coordinates of A are obtained by solving $y = m_1 x + b$ and $y = m_2 x + e$

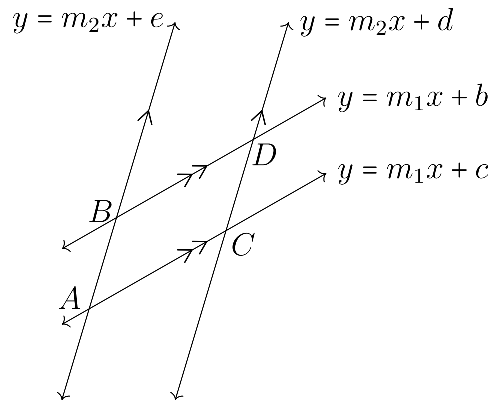


Figure 35: Parallelogram ABCD

to obtain

$$A(x_1, y_1) = \left(\frac{b-e}{m_2 - m_1}, \frac{m_2 b - m_1 e}{m_2 - m_1}\right)$$

The coordinates of B are obtained by solving $y = m_1 x + c$ and $y = m_2 x + e$ to obtain

$$B(x_2, y_2) = \left(\frac{c-e}{m_2 - m_1}, \frac{m_2 c - m_1 e}{m_2 - m_1}\right)$$

Then,

$$(x_2 - x_1) = \frac{c-b}{m_2 - m_1} \text{ and } y_2 - y_1 = \frac{m_2(c-b)}{m_2 - m_1}$$

making,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{c-b}{m_2 - m_1}\sqrt{1 + m_2^2}$$

Since the line equations to be solved to find CD are the same as those we just used to find AB except for the replacement of e with d then we also have,

$$CD = \frac{c-b}{m_2 - m_1}\sqrt{1 + m_2^2}$$

So $AB = CD$.
Similar calculations give.

$$AC = BD = \frac{c-b}{m_2 - m_1}\sqrt{1 + m_1^2}$$

We have proved the opposite sides of a parallelogram are equal in length. □

Chapter 7

Basic Proportionality Theorem and Converse

7.1 Basic Proportionality Theorem

This is the second key theorem to strengthening the foundations of plane geometry.The theorem is due to Thales, another ancient Greek mathematician.

Theorem 16. *(Basic Proportionality - Thales) If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides those two sides in the same ratio.*

Proof. Our proof refers to Figure 36. We start with the triangle $\triangle ABC$ and draw a

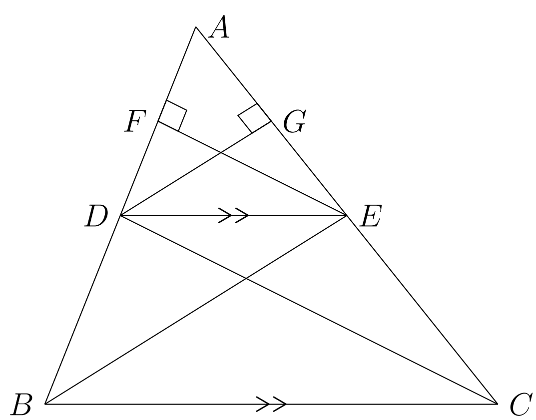


Figure 36:

line parallel to BC that intersects AB and AC at D and E respectively.
We want to prove $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

We draw DG perpendicular to AC and EF perpendicular to AB.
Now EF is the perpendicular height of both of the triangles $\triangle ADE$ and $\triangle DBE$.
Hence,

$$Area \triangle ADE = \frac{AD \times EF}{2}$$
$$Area \triangle DBE = \frac{DB \times EF}{2}$$

Yielding,

$$\frac{Area \triangle ADE}{Area \triangle DBE} = \frac{AD}{DB} \tag{7.1.1}$$

Similarly DG is the perpendicular height of both of the triangles $\triangle ADE$ and $\triangle DEC$.
Hence,

$$Area \triangle ADE = \frac{AE \times DG}{2}$$
$$Area \triangle DEC = \frac{EC \times DG}{2}$$

Yielding,

$$\frac{Area \triangle ADE}{Area \triangle DEC} = \frac{AE}{EC} \tag{7.1.2}$$

Now,

$$Area \triangle DBE = Area \triangle DEC$$

since both triangles are on the same base and between the same parallel lines and by Theorem 14 on page 27, the perpendicular distance between two parallel lines is the same wherever it is drawn, giving these two triangles the same perpendicular height or altitude as well as the same base DE.
Hence, from (7.1.1) and (7.1.2) we have,

$$\frac{Area \triangle ADE}{Area \triangle DBE} = \frac{Area \triangle ADE}{Area \triangle DEC} \implies \frac{AD}{DB} = \frac{AE}{EC}$$

□

7.2 Converse of Basic Proportionality Theorem

Theorem 17. *If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side,* *Proof.* Our proof refers to Figure 37. We position points D on AB and E on AC such that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$. If DE is not parallel to BC then another line beginning at D must be, so we draw DF as such a parallel line and mark DF and BC accordingly with the parallel notation.

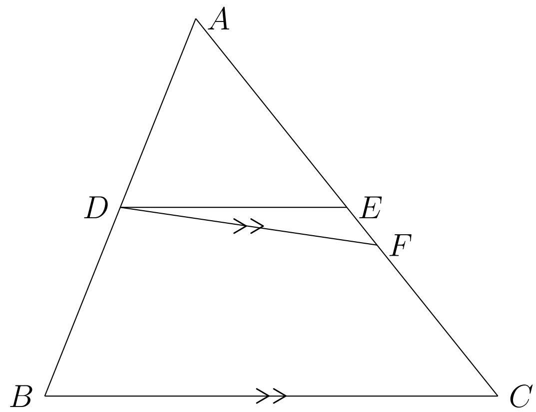


Figure 37:

Given $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, we want to prove $DE \parallel BC$.

Now, by the Basic Proportionality Theorem 16 on page 30 , if $DF \parallel BC$ then

$$\frac{AD}{DB} = \frac{AF}{FC}$$

We are given $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, so combining these two expressions for $\dfrac{AD}{DB}$ we have,

$$\begin{aligned}
\frac{AF}{FC} &= \frac{AE}{EC} \\
\implies \frac{AF}{FC} + 1 &= \frac{AE}{EC} + 1 \\
\implies \frac{AF + FC}{FC} &= \frac{AE + EC}{EC} \\
\implies \frac{AC}{FC} &= \frac{AC}{EC} \\
\implies FC &= EC
\end{aligned}$$

But this is only possible when F and E are the same point. So DF is the line DE itself. We conclude $DE \parallel BC$. □

Chapter 8

Congruence and Similarity

8.1 Similarity in general

Definition 18. *Similarity*
When two figures have exactly the same shape, they are similar. They may not be the same size, but if so, the smaller has been magnified to produce the larger, preserving angles but stretching lines by the same factor.

Notation 12. *If Figure 1 is similar to Figure 2 we write* $Fig1 \sim Fig2$.

8.2 Similar Triangles

Definition 19. *Similar Triangles*
Two triangles are similar if and only if:

1. *all pairs of corresponding angles are congruent.*
2. *all pairs of corresponding sides are proportional.*

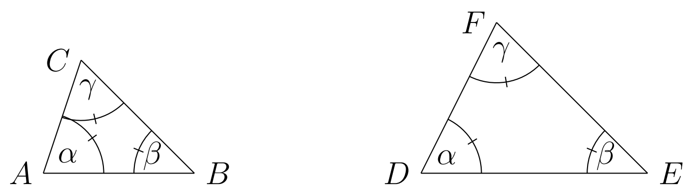


Figure 38: Similar Triangles

So, referring to Figure 38, if $\triangle ABC \sim \triangle DEF$ then we must have;

1. $\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$
2. $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

8.3 Congruence in general

Definition 20. *Congruence*
When two figures have the same shape (~) and all corresponding lines and angles have equal measure (=), the two figures are congruent. This means that one is the mirror image of the other, or that either object can be repositioned so as to coincide precisely with the other object.

Notation 13. *If two figures, 1 and 2, are congruent, we use a symbol that is a combination of similar and equal and we write* $Fig1 \cong Fig2$.

8.4 Congruent Angles

Definition 21. *Congruent Angles*
An angle consists of four elements:

1. *A vertex*
2. *An initial ray*
3. *A terminal ray*
4. *A degree of rotation about the origin of the terminal ray from the initial ray.*

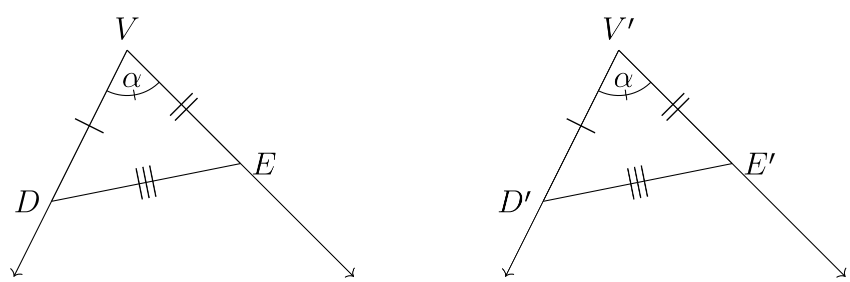


Figure 39: Congruent Angles

Accordingly, two angles are congruent if we have:

1. *The vertex of the second placed on the vertex of the first.*
2. *The initial ray of the second covering the initial ray of the first.*
3. *The terminal ray of the second covering the terminal ray of the first.*
4. *The degree of rotation of the second the same as the degree of rotation of the first, which requires the distances between two corresponding points on the initial and terminal rays remaining the same. For example,* $DE = D'E'$ *on Figure 39 if* D' *maps to* D *and* E' *to* E.

These conditions are shown in Figure 39.

8.5 Congruent Triangles

Definition 22. *Congruent Triangles*
Two triangles are congruent if and only if:

1. *all pairs of corresponding angles are congruent.*
2. *all pairs of corresponding sides have equal length and also the same position with respect to the angles.*

So, referring to Figure 40, if $\triangle ABC \cong \triangle DEF$ *then we must have;*

1. $\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$
2. $AB \cong DE; BC \cong EF; AC \cong DF$

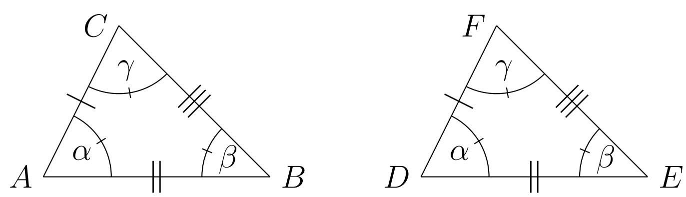


Figure 40: Congruent Triangles

8.6 Theorems for Similar Triangles

We first consider criteria for two triangles to be similar if we are provided with only some of the congruences and equalities listed in Definition 19 on page 33. In what follows the symbol A means an angle and the symbol S a side. Accordingly we will prove that two triangles are similar if they obey any one of the following rules.

1. $SAS \sim$: If an angle of one triangle is congruent to an angle of the other triangle and the pairs of corresponding sides forming the angle are proportional then the two triangles are similar.
2. $AAA \sim$: If all three corresponding angles in each triangle are congruent then the two triangles are similar.
3. $SSS \sim$: If the ratios of all three pairs of corresponding sides are equal the two triangles are similar.

In each case we need to prove whatever is missing from all six of the conguences listed in Definition 19 on page 33.

8.7 Theorems for Congruent Triangles

We will then consider criteria for two triangles to be congruent if we are provided with only some of the congruences listed in Definition 22 on page 35. In what follows the symbol A again means an angle and the symbol S a side. Accordingly we will prove that two triangles are congruent if they obey any one of the following rules.

1. $SAS \cong$: If an angle of one triangle is congruent to an angle of a second triangle and the pairs of corresponding sides that form the angles are congruent then the two triangles are congruent.

2. $SSS \cong$: If all three pairs of corresponding sides of two triangles are congruent then the two triangles are congruent.

3. $ASA \cong$: If two corresponding pairs of angles of the two triangles are congruent and the corresponding sides adjacent to these angles are congruent then the two triangles are congruent.

4. $AAS \cong$: If two angles and a non-included side for one triangle are congruent to two angles and the corresponding non-included side of the other triangle then the two triangles are congruent.

In each case we need to prove whatever is missing from all six of the conguences and equalities listed in Definition 22 on page 35.

Chapter 9

Similarity and Congruence Triangle Theorems

9.1 Order of Proofs

We will now prove the three similarity and four congruence theorems for a pair of triangles. We will prove these theorems using only the results we have already developed. The seven theorems will be proved in the following order with the results of a prior theorem being used in the proof of a current theorem. The order will be:

1. SAS similarity or $SAS \sim$
2. AAA similarity or $AAA \sim$
3. SAS congruence or $SAS \cong$
4. SSS congruence or $SSS \cong$
5. ASA congruence or $ASA \cong$
6. SSS similarity or $SSS \sim$
7. AAS congruence or $AAS \cong$

Note: We note that $AAA \sim$ is the same as $AA \sim$ since if two pairs of corresponding angles of two triangles are congruent (and specifically equal in measure), then the third pair must also be congruent to each other since the measures of each of the three angles add to $180°$.

9.2 Proofs Part A.

Theorem 18. *(SAS similarity): Two triangles are similar if an angle of one triangle is congruent to an angle of the other triangle and the corresponding pairs of sides forming these angles have the same ratio, that is, are proportional.*

Proof. Consider two triangles, $\triangle ABC$ and $\triangle DEF$, on the Cartesian Plane where, in Figure 41, we have omitted the axes and gridlines.

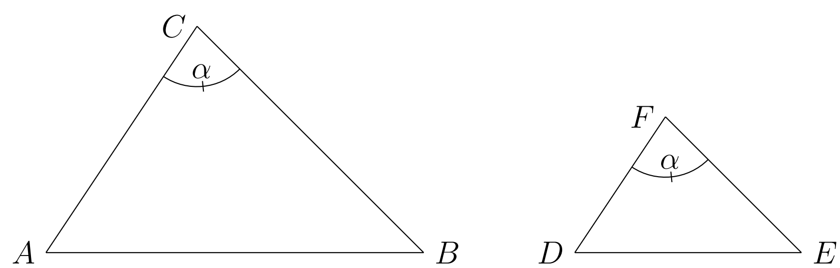


Figure 41: SAS Similarity 1

We are given $\angle C \cong \angle F$ with measure α, say, and $\dfrac{AC}{DF} = \dfrac{BC}{EF}$. We need to prove

$$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \frac{AB}{DE} = \frac{AC}{DF} \ \mathit{or}\ \frac{AB}{DE} = \frac{BC}{EF}. \tag{9.2.1}$$

Referring to Definition 21 of congruent angles on page 34, if we overlay $\angle DFE$ on $\angle ACB$, we have the sides congruences as shown in Figure 42 with F overlaying C, D overlaying a point P on AC and E overlaying a point Q on BC.

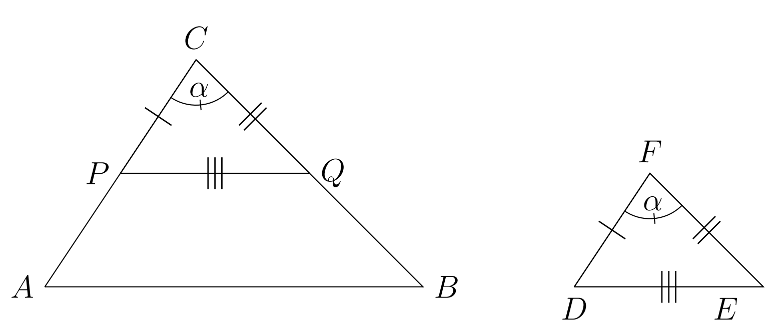


Figure 42: SAS Similarity 2

Then with $DF = CP$ and $EF = CQ$, the given equality $\dfrac{AC}{DF} = \dfrac{BC}{EF}$ becomes,

$$\begin{aligned} \frac{AC}{CP} &= \frac{BC}{CQ} \\ \implies \frac{AC}{CP} - 1 &= \frac{BC}{CQ} - 1 \\ \implies \frac{AC - CP}{CP} &= \frac{BC - CQ}{CQ} \\ \implies \frac{AP}{CP} &= \frac{BQ}{CQ} \end{aligned}$$

By Thales Basic Proportionality Theorem 16 on page 30, this gives $PQ \parallel AB$.
By the Corresponding Angles Theorem 2 on page 10, we then have $\angle CPQ \cong \angle BAC$ both with measure β, and $\angle CQP \cong \angle ABC$ both with measure γ, say.
That takes care of the angles. Next we find the point R on AB such that $QR \parallel AC$ and draw QR. We then have Figure 43.

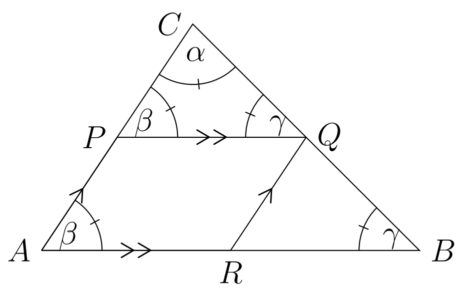


Figure 43: SAS similarity 3

By the Converse Theorem 17 to Thales' Theorem on page 31, since $QR \parallel AC$ we have,

$$\begin{aligned} \frac{BR}{AR} &= \frac{BQ}{CQ} \\ \frac{BR}{AR} + 1 &= \frac{BQ}{CQ} + 1 \\ \frac{BR + AR}{AR} &= \frac{BQ + CQ}{CQ} \\ \frac{AB}{AR} &= \frac{BC}{CQ} \end{aligned}$$

But $ARQP$ is a parallelogram and by Theorem 15 on page 29 the opposite sides $AR = PQ$, so that we now have $\dfrac{AB}{PQ} = \dfrac{BC}{CQ}$.

But $PQ = DE$ and $CQ = EF$ so that, with (9.2.1) included,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

and we are done. We conclude $\triangle ABC \sim \triangle DEF$ and the $SAS \sim$ theorem is proved. $\square$

Theorem 19. *(AAA similarity): Two triangles are similar if the pairs of corresponding angles are all equal.*

Proof. As shown in Figure 44, we are given $\triangle ABC$ and $\triangle DEF$ with $m\angle A = m\angle D = \alpha$, $m\angle B = m\angle E = \beta$, $m\angle C = m\angle F = \gamma$. We are required to prove the ratios of the pairs of corresponding sides are equal, that is,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Construct P and Q such that $AP = DE$ and $AQ = DF$ as on Figure 44.

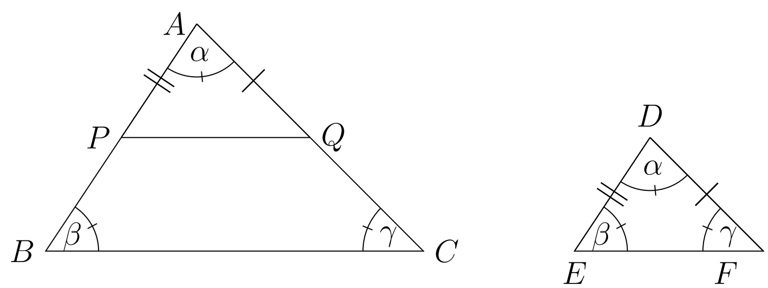


Figure 44: AAA similarity

Then with two corresponding sides equal and the angle $\angle A \cong \angle D$, in common, by the SAS~ Theorem 18 on page 38 we conclude $\triangle APQ \sim \triangle DEF$, and in particular, $m\angle APQ = \beta$ and $m\angle AQP = \gamma$.
Then by the Converse to the Corresponding Angles Theorem 11 on page 20, $PQ \parallel BC$. So by Thales' Basic Proportionality Theorem 16 on page 30,

$$\begin{aligned} \frac{BP}{AP} &= \frac{QC}{AQ} \\ \implies \frac{BP}{AP} + 1 &= \frac{QC}{AQ} + 1 \\ \implies \frac{AP + PB}{AP} &= \frac{AQ + QC}{AQ} \\ \implies \frac{AB}{AP} &= \frac{AC}{AQ} \\ \implies \frac{AB}{DE} &= \frac{AC}{DF} \end{aligned}$$

since $DE = AP$ and $DF = AQ$ by construction.
Then by the previous $SAS \sim$ Theorem 18 on page 38 , with $m\angle C = m\angle D = \alpha$ in the two triangles and the corresponding pairs of the sides forming these angles in the same ratio, we conclude $\triangle ABC \sim \triangle DEF$. □

Theorem 20. *(SAS congruence)*
If an angle of one triangle is congruent to an angle of a second triangle and the pairs of corresponding sides that form the angles are congruent then the two triangles are congruent.

Proof. As depicted in Figure 45, we are given triangles $\triangle ABC$ and $\triangle DEF$ with $AB \cong DE$, $BC \cong EF$ and $m\angle B = m\angle E = \alpha$.

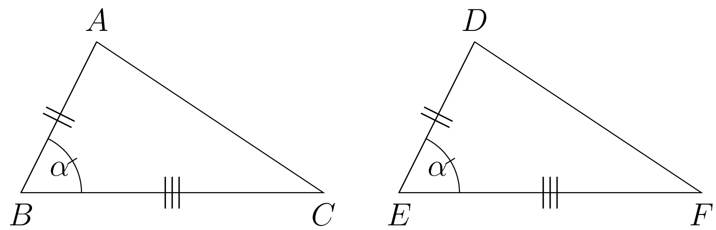


Figure 45: SAS congruence

We need to prove $AC \cong DF$, $\angle A \cong \angle D$ and $\angle C \cong \angle F$.
First the angles. We note $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ since both ratios are equal to 1. So by SAS~ for similarity, Theorem 18 on page 38, the two triangles are similar and therefore the corresponding angles are equal.
Thus, $\triangle ABC \sim \triangle DEF \implies \angle A \cong \angle D$ and $\angle C \cong \angle F$.
Second, again by SAS~ for similar triangles, the three ratios of corresponding sides are equal, that is,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

But we are given $AB = DE \implies \dfrac{AB}{DE} = 1$ and therefore $\dfrac{AC}{DF} = 1 \implies AC \cong DF$.
□

9.3 Interlude

We need the following theorem concerning an isosceles triangle. Its proof depends on a previous theorem.

Theorem 21. *The base angles of an isosceles triangle are congruent.*

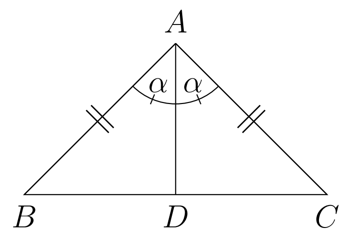


Figure 46: Isosceles Triangle

Proof. We need to show $\angle B \cong \angle C$.
Figure 46 shows an isosceles triangle (two equal sides) with the bisector of the angle $\angle A$ drawn in and projected to meet the base line BC at a point D.
Then $\dfrac{AB}{AC} = \dfrac{AD}{AD} = 1$ and the included angles are equal, so by SAS~, Theorem 18 on page 38 we have similar triangles, $\triangle ABD \sim \triangle ADC$, and conclude $\angle B \cong \angle C$. $\square$

9.4 Proofs Part B

Theorem 22. *(Jacobs) SSS congruence: If the three corresponding pairs of sides of two triangles are equal in length then the two triangles are congruent.*

Proof. Consider Figure 47.

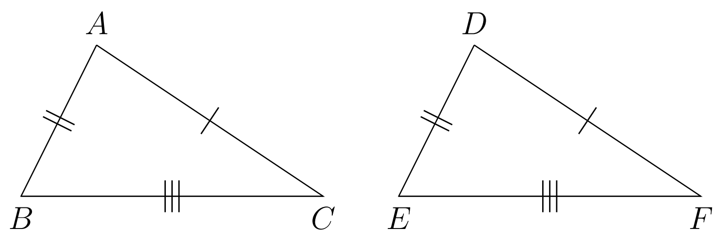


Figure 47: SAS congruence 1

Given the side equalities $AB \cong DE$, $AC \cong DF$ and $BC \cong EF$, we need to prove $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.
We construct the ray $\overrightarrow{BX}$ such that $m\angle CBX = m\angle E = \alpha$.
We position the point P on $\overrightarrow{BX}$ such that $BP = DE$.
We join C to P and A to P giving Figure 48.
By $SAS \sim$, Theorem 18 on page 38, $\triangle BPC \sim \triangle DEF$.
We now want to prove $\triangle BPC \sim \triangle ABC$ which would make all three triangles similar

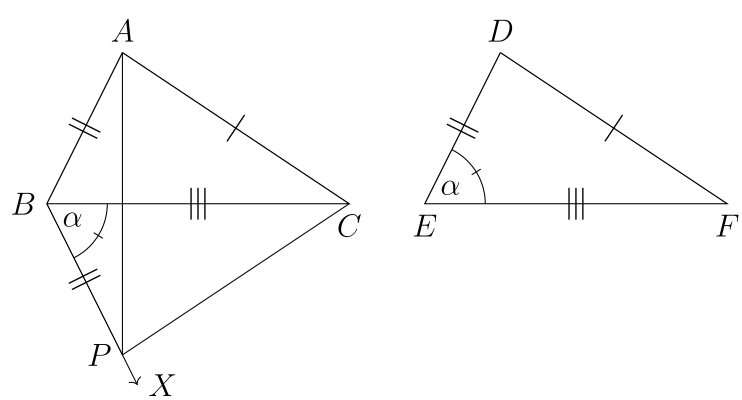


Figure 48: SAS congruence 2

and in particular give the result we seek, namely, $\triangle ABC \sim \triangle DEF$.
Now $\triangle BPC \sim \triangle DEF$ gives us, $\dfrac{BP}{DE} = \dfrac{BC}{EF} = \dfrac{PC}{DF}$. But $BP = DE$ making $\dfrac{BP}{DE} = 1$ and hence $\dfrac{PC}{DF} = 1 \implies PC = DF$.

We are given $AC = DF$ so $PC = AC$.
Therefore $\triangle APC$ is an isosceles triangle having two equal sides AC and PC.
For the same reason $\triangle ABP$ is also an isosceles triangle having $AB = BP$.
By Theorem 21 on page 42 we can therefore label the two pairs of equal base angles with β and γ respectively as shown in Figure 49.

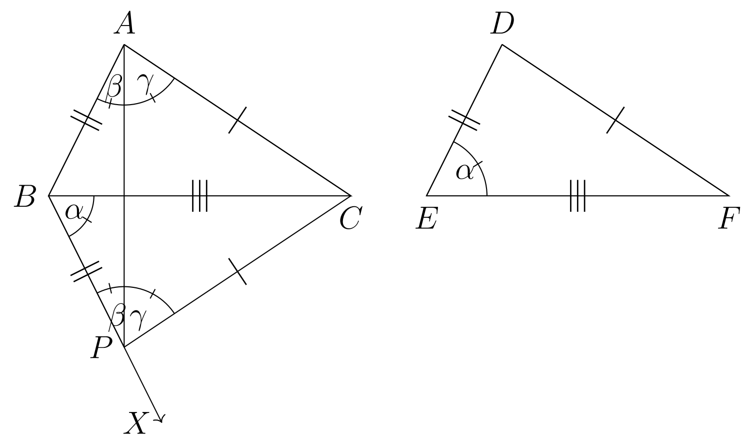


Figure 49: SAS congruence 3

Then $m\angle BAC = m\angle BPC = \beta + \gamma$, and hence $\triangle ABC \sim \triangle BPC$ by the $SAS \sim$ Theorem 18 on page 38. Since we also showed $\triangle BPC \sim \triangle DEF$ then the three corresponding angles in these three triangles are all equal and in particular we not only have equality of corresponding sides in $\triangle ABC$ and $\triangle DEF$ but we also have equality of corresponding angles, namely,

$$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$$

We have proved $\triangle ABC \cong \triangle DEF$ and the $SSS \cong$ theorem in general. □

Theorem 23. *(ASA Congruence) If two angles of one triangle are congruent to two angles of another triangle and the corresponding sides common to both these angles are congruent then the triangles are congruent.*

Proof. Consider Figure 50.

Given $m\angle B = m\angle E = \gamma$ and $m\angle C = m\angle F = \beta$ and $BC \cong EF$ we need to prove $AB \cong DE$, $AC \cong DF$ and $\angle A \cong \angle D$.

The third angles $\angle A$ and $\angle D$ must also be congruent since both measure $180° - \alpha - \beta$ so we have satisfied the conditions for $AAA \sim$ and have $\triangle ABC \sim \triangle DEF$. This means the corresponding sides have the same ratio, that is,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

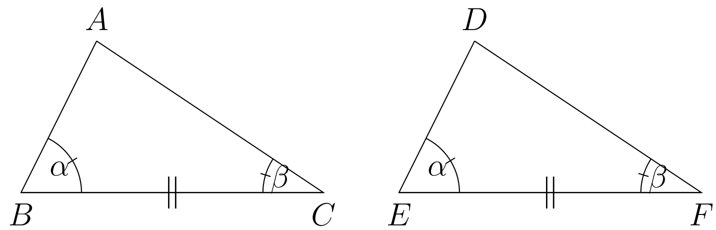


Figure 50: ASA congruence

But we are given $BC \cong EF$ and so their ratio is 1. Then the other two side ratios are also 1 and we have

$$AB \cong DE,\ AC \cong DF$$

With three angles and three sides of $\triangle ABC$ congruent to three sides and three angles of $\triangle DEF$, we conclude $\triangle ABC \cong \triangle DEF$ and the $ASA \cong$ theorem is proved. □

Theorem 24. *(SSS similarity) If the three pairs of corresponding sides of two triangles are proportional (have the same ratio), then the two triangles are similar.*

Proof. Consider Figure 51.

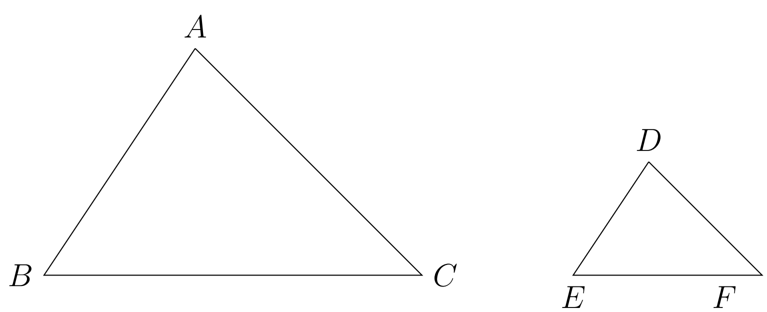


Figure 51: SSS similarity 1

We are given

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} \tag{9.4.1}$$

We want to prove $\triangle ABC \sim \triangle DEF$ so we need to prove,

$$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$$

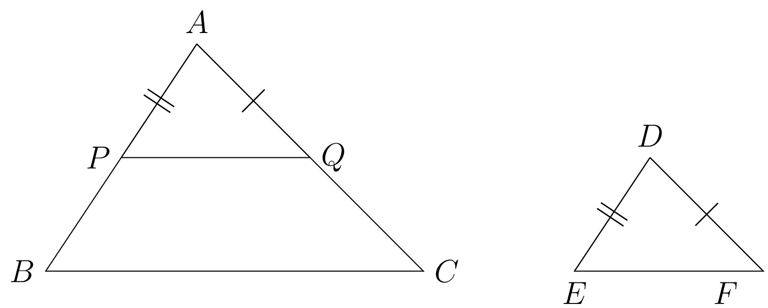


Figure 52: SSS similarity 2

We construct PQ such that $AP = DE$ and $AQ = DF$. See Figure 52.

We are given in (9.4.1) that,

$$
\begin{aligned}
&\frac{AB}{DE} = \frac{AC}{DF} \\
\implies &\frac{AB}{AP} = \frac{AC}{AQ} \text{ since } AP = DE,\ AQ = DF \\
\implies &\frac{AB}{AP} - 1 = \frac{AC}{AQ} - 1 \\
\implies &\frac{AB - AP}{AP} = \frac{AC - AQ}{AQ} \\
\implies &\frac{BP}{AP} = \frac{CQ}{AQ}
\end{aligned}
$$

Then, by the converse to Thales' Basic Proportionality Theorem 17 on page 31 we have $PQ \parallel BC$. So, by the Corresponding Angles Theorem 2 on page 10 the angle measures are,

$$m\angle APQ = m\angle ABC = \beta,\ m\angle AQP = m\angle ACB = \gamma \text{ say}$$

and the third pair of triangle angles $\angle A$ and $\angle D$ must also be equal in measure. By $AAA \sim$ we conclude,

$$\triangle ABC \sim \triangle APQ$$

But this means the ratio of pairs of corresponding sides are equal, or,

$$
\begin{aligned}
&\frac{AB}{AP} = \frac{AC}{AQ} = \frac{BC}{PQ} \\
\implies &\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{PQ} \qquad (9.4.2)
\end{aligned}
$$

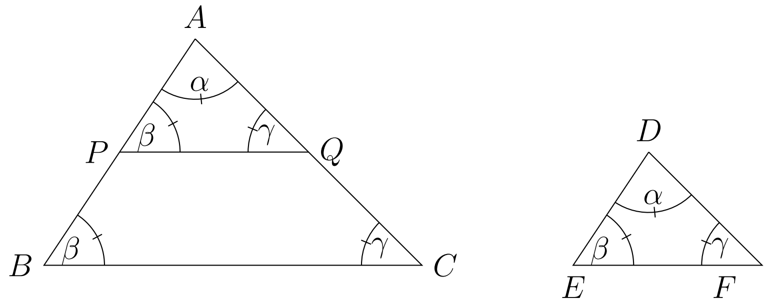


Figure 53: SSS similarity 3

since $AP \cong DE$ and $AQ \cong DF$. From Equations (9.4.1) and (9.4.2) we conclude,

$$\frac{BC}{PQ} = \frac{BC}{EF} \tag{9.4.3}$$

and that $PQ = EF$. So $\triangle APQ$ and $\triangle DEF$ have three pairs of corresponding congruent sides. By $SSS \cong$ proved in Theorem 22 on page 43, we have $\triangle APQ \cong \triangle DEF$ so we can label all angles and sides as in Figure 53. Since all three angles are congruent and the pairs of corresponding sides are in the same ratio, as given by Equations (9.4.2) and (9.4.3), namely,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF},$$

we conclude $\triangle ABC \sim \triangle DEF$. □

Theorem 25. *(AAS congruency) If two angles and a non-included side of one triangle are congruent to two angles and the corresponding non-included side of another triangle, then the triangles are congruent.*

Proof.

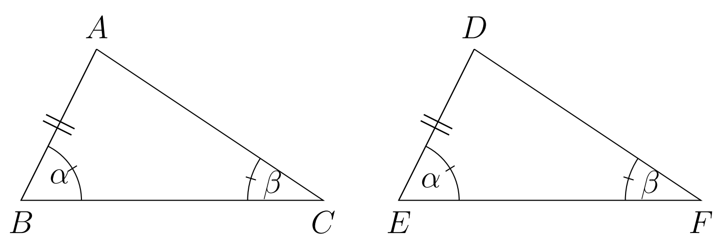


Figure 54: AAS congruence

Consider Figure 54. We are given $m\angle B = m\angle E = \alpha$ say, and $m\angle C = m\angle F = \beta$ say, and also $AB \cong DE$. The third angles $\angle A$ and $\angle D$ in the two triangles must also be

congruent since both measure $180° - \alpha - \beta$.
Then by Theorem 19 on page 40, $AAA \sim$, we have $\triangle ABC \sim \triangle DEF$ and specifically,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

But $AB \cong DE$ and so $\dfrac{AB}{DE} = 1$. In turn $AC \cong DF$ and $BC \cong EF$. Given three pairs of congruent angles and sides we conclude $\triangle ABC \cong \triangle DEF$. □

We could have also invoked $ASA \cong$ proved as Theorem 23 immediately after finding the three pairs of angles were congruent as well as the given $AB = DE$.

9.5 Right Triangles Special Case

If two triangles are both right triangles then there is a further congruence theorem labelled RHS.

Theorem 26. *(RHS≅) If two right triangles have their hypotenuses and one other side equal then the triangles are congruent.*

Proof. Consider Figure 55. We are given $AC \cong DF$, $BC \cong EF$, and $m\angle B = m\angle E = 90°$.

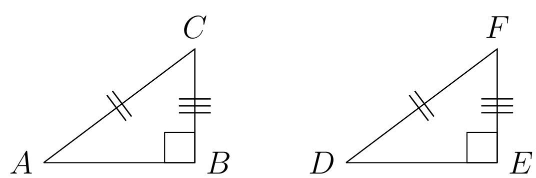


Figure 55: RHS Congruence

We need to prove $AB \cong DE$.
By the Pythagorean Theorem we have:

$$AB^2 = AC^2 - BC^2$$
$$DE^2 = DF^2 - EF^2$$

We are given $AC \cong DF$ and $BC \cong EF$ so $AB \cong DE$.
By the $SSS \cong$ Theorem 22 on page 43, since we have three pairs of congruent sides, we conclude $\triangle ABC \cong \triangle DEF$. □

Chapter 10

Euclid and Circles

Let us complete our use of Euclid's postulates. Euclid's fourth postulate is:

Given any straight line segment, a circle can be drawn having the segment as radius and one end of the segment as center.

The following theorem does not directly require any of the similarity or congruence theorems but it does need the isosceles angle Theorem 21 proved on page 42 which in turn required the SAS similarity Theorem 18 proved on page 38. You will find almost all proofs of theorems in plane geometry require one or more of the parallel lines, congruence and similarity theorems either directly or indirectly. Here is an example prompted by Euclid's Fourth.

Theorem 27. *The angle at the center of a circle is twice the angle at the circumference when both are subtended by the same arc.*

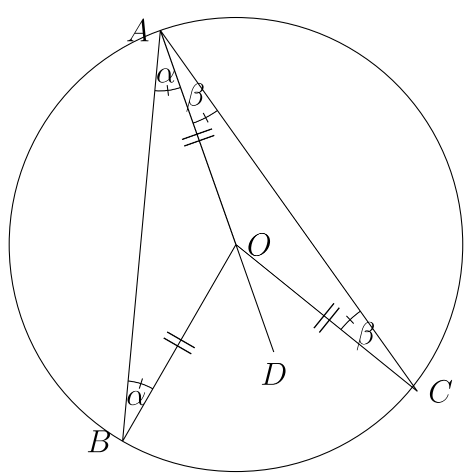


Figure 56: Circle Angles

Proof. We choose any arc $\overset{\frown}{BC}$ on the circumference of a circle with center O. We join the ends of the arc to O and we say the arc subtends the angle $\angle BOC$ at the center O of the circle. We also draw lines from B and C to any point A on the circumference. We say the arc $\overset{\frown}{BC}$ subtends the angle $\angle BAC$ at the circumference of the circle. Our theorem is that $m\angle BOC = 2 \times m\angle BAC$.

We also draw the line AO and extend it to an arbitrary point D to create two angles $\angle BOD$ and $\angle COD$ which are the exterior angles of the triangles $\triangle ABO$ and $\triangle ACO$. Both of these triangles have radii as sides so they are isosceles and by Theorem 21 on page 42 their pairs of base angles can be labelled α and β as shown.

This shows, $\angle BAC = \alpha + \beta$.

By the Exterior Angles Theorem 6 on page 16, $m\angle BOD = 2\alpha$ and $m\angle COD = 2\beta$ so that,

$$m\angle BOC = 2\alpha + 2\beta = 2 \times m\angle BAC$$

and our theorem is proved. □

Chapter 11

Exercises

11.1 Proven Theorems

The foundation of plane geometry is simply Euclid's five postulates, particularly the fifth. We proved the following are 14 consequential theorems/converses and not just postulates:

1. The alternate interior angles formed by a transversal intersecting two parallel lines are equal in measure.

2. (Converse) If two lines are intersected by a transversal and the alternate interior angles are equal then the two lines are parallel.

3. When a transversal intersects two parallel lines, the pairs of corresponding angles are congruent.

4. (Converse) When a transversal line cuts two other lines, if the corresponding angles formed by the transversal and each of the other lines are equal then the two lines are parallel.

5. When a transversal intersects two parallel lines, the measures of the pairs of same side interior angles add to $180°$.

6. (Converse) If two lines are intersected by a transversal and the same side interior angles are supplementary (add to $180°$ in measure) then the two lines are parallel.

7. $SAS \sim$: If an angle of one triangle is congruent to an angle of the other triangle and the pairs of corresponding sides forming the angle are proportional then the two triangles are similar.

8. $AAA \sim$: If all three corresponding angles in each triangle are congruent then the two triangles are similar.

9. $SSS \sim$: If the ratios of all three pairs of corresponding sides are equal the two triangles are similar.

10. $SAS \cong$: If an angle of one triangle is congruent to an angle of a second triangle and the pairs of corresponding sides that form the angles are congruent then the two triangles are congruent.

11. $SSS \cong$: If all three pairs of corresponding sides of two triangles are congruent then the two triangles are congruent.

12. $ASA \cong$: If two corresponding pairs of angles of the two triangles are congruent and the corresponding sides adjacent to these angles are congruent then the two triangles are congruent.

13. $AAS \cong$: If two angles and a non-included side for one triangle are congruent to two angles and the corresponding non-included side of the other triangle then the two triangles are congruent.

14. $RHS \cong$: If two right triangles have their hypotenuses and one other side equal then the triangles are congruent.

In the course of proving these theorems we proved the following theorems, converses or facts for triangles:

15. The measures of the interior angles of a triangle sum to 180°.

16. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

17. (Isosceles Triangles) If two sides of a triangle are congruent then the (base) angles opposite these sides are congruent.

18. (Converse) If the two base angles of a triangle are congruent then the sides opposite these angles are congruent.

19. (Thales) If a line is drawn parallel to one side of a triangle intersecting the other two sides then it divides those two sides in the same ratio.

20. (Converse) If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

21. (Pythagoras) In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

22. The area of a triangle is $\frac{1}{2}\text{base} \times$ perpendicular height.

We also proved these theorems:

23. When two lines in a plane intersect one another, the two pairs of opposite angles are congruent.

24. The shortest distance from a point to a line is the perpendicular line segment from the point to the line.

25. The perpendicular distance between two parallel lines is constant.

26. The opposite sides of a parallelogram are congruent (equal in length).

11.2 More Triangle Theorems

The reader is invited to prove the following common theorems regarding triangles. You can use any of the theorems 1 to 26 proved above and/or you can use any theorem you have already proved from the list below. For example, if you have proved, say, 31 below then you can use it to prove 32 if it applies. Hints are provided after the list of theorems (exercises). The standard triangle referred to in the hints is labelled thus:

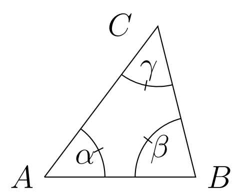


Figure 57: Standard Triangle

11.3 Example

It is recommended that you format your proofs like this:

- State any definition referred to.
- List what you are given as equations and/or information added to a diagram.
- State what you are required to prove.
- Give the proof, stating the reasons for each statement, which may be a prior theorem.

As an example, here is the proof of Theorem 27 below.

Theorem: The measure of each angle of an equilateral triangle is $60°$.

Proof.
Definition: An equilateral triangle has three equal sides.
Given:

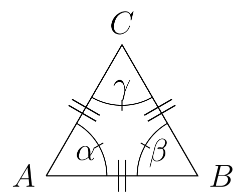


Figure 58: Equilateral Triangle

Required to prove: $\alpha = \beta = \gamma = 60°$.
Now an equilateral triangle is also an isosceles triangle where any side may be taken as the base.
By Theorem 17 with AB as base, $\alpha = \beta$.
By Theorem 17 with BC as base, $\beta = \gamma$.
Hence $\alpha = \beta = \gamma$.
But by Theorem 15, $\alpha + \beta + \gamma = 180°$.
Therefore $\alpha = \beta = \gamma = 60°$. □

11.4 Exercises

27. The measure of each angle of an equilateral triangle is $60°$.

28. A triangle may have at most one right or one obtuse angle.

29. The acute angles of a right triangle are complementary.

30. If two angles of a triangle are congruent to two angles of another triangle then the remaining pair of angles are also congruent.

31. The median of an isosceles triangle bisects the vertex angle. (Note: a median joins a vertex to the midpoint of the opposite side.)

32. In a right triangle, the altitude from the vertex separates the original triangle into two new triangles similar to the original triangle and to each other.

33. If two triangles are similar then their corresponding altitudes have the same ratio as the pairs of corresponding sides.

34. If two triangles are similar then their corresponding angle bisectors have the same ratio as the pairs of corresponding sides.

35. If two triangles are similar then their corresponding medians have the same ratio as the pairs of corresponding sides.

36. If a point is on the bisector of an angle then it is equidistant from the two sides forming the angle.

37. (Converse) If a point is equidistant from the two sides forming an angle then it lies on the bisector of that angle.

38. An angle bisector of a triangle divides the opposite side into two segments with the same ratio as the other two sides.

39. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

40. If a point is on the perpendicular bisector of a segment then it is equidistant from the end points of the segment.

41. (Converse to previous) If a point is equidistant from the end points of a segment then it is on the perpendicular bisector of the segment.

42. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and one-half its length.

43. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.

44. On the standard $\triangle ABC$ if D and E are the midpoints of AC and BC, and F is the point of intersection of AE and BD, show triangles $\triangle DEF$ and $\triangle BAF$ are similar and that the length of each side of $\triangle BAF$ is twice as long as the corresponding side of $\triangle DEF$.

45. On the standard $\triangle ABC$, with D a point on AC and E a point on BC such that $DE \parallel AB$, the triangles $\triangle ABC$ and $\triangle CDE$ are similar.

46. (Mid-segment theorem) On the standard $\triangle ABC$ if D and E are the midpoints of AC and BC, then $DE \parallel AB$ and $DE = \frac{1}{2}AB$.

47. In a triangle, the angle opposite the longest side is greater than the angle opposite the shortest side.

48. In a triangle, prove the side opposite the greatest angle is longer than the side opposite the least angle.

49. (Triangle Inequality Theorem) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

50. In a right triangle $\triangle ABC$ with C a right angle and CD the altitude from C to AB we have the following (mean proportional)equations:

$$\frac{AD}{AC} = \frac{AC}{AB}; \frac{BD}{BC} = \frac{BC}{AB}; \frac{AD}{CD} = \frac{CD}{DB}$$

51. (Pythagoras) Prove the Pythagorean Theorem using the previous theorem.

52. The length of the median drawn to the hypotenuse of a right triangle is one-half the length of the hypotenuse.

53. The medians of a triangle meet at a point called the centroid.

54. An equilateral triangle can be divided into nine congruent triangles.

55. A line through the centroid of an equilateral triangle parallel to the base divides the area of the triangle in a 5:4 ratio.

56. The angle bisectors of a triangle meet at a point called the incenter.

57. The perpendicular bisectors of a triangle meet at a point called the circumcenter.

58. The altitudes of a triangle meet at a point called the orthocenter.

59. The centroid, circumcenter and orthocenter of a triangle are collinear on a line called the Euler line.

60. (Ceva's theorem) Given a triangle $\triangle ABC$, let the lines AD, BD and CD be drawn from the vertices to a common point D (not on one of the sides of $\triangle ABC$, to meet opposite sides at F, E and G respectively. (The segments AF, BE, and CG are known as cevians.) Then,

$$\frac{AG}{GB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = 1.$$

61. (Napoleon's theorem) If equilateral triangles are constructed on the sides of any triangle, either all outward or all inward, the lines connecting the centers (centroids) of those equilateral triangles themselves form an equilateral triangle.

62. (Viviani's theorem) The sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude.

63. (Pompeui's theorem) Given an equilateral triangle $\triangle ABC$ and a point P in the plane of the triangle $\triangle ABC$, the lengths PA, PB, and PC form the sides of a (maybe, degenerate, that is straight line) triangle.

64. (Apolloniu's Theorem) The sum of the squares of any two sides of a triangle is equal to twice its square on half of the third side plus twice its square on the median bisecting the third side.
(If D is the midpoint of AB then $CA^2 + CB^2 = 2(AD^2 + CD^2)$)

65. (Morley's Trisector theorem) In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.

11.5 Hints for Solving the Exercises

28. Let $\alpha = 90°$ and $\beta = 90°$. Find γ using Theorem 15. Similarly for two obtuse angles.

29. Let $\alpha = 90°$. Use Theorem 15.

30. In two standard triangles let the angles be 1,2,3 and 4,5,6 respectively. Use Theorem 15 given 1=4 and 2=5.

31. Draw the median from A to BC and use $SAS \cong$.

32. Let $\alpha = 90°$ in standard triangle. Draw altitude from A to BC. Use Theorem 15 to label the other two angles as β and γ. Use $AAA \sim$ theorem on all three pairs of corresponding sides.

33. Draw two similar triangles $\triangle ABC$ and $\triangle DEF$, with altitudes AX and DY from A to BC and D to EF. Label any congruent angles, use $AAA \sim$ on $\triangle ACX$ and $\triangle DFY$ and write ratios for corresponding sides.

34. As for preceding theorem with angle bisectors replacing altitudes and using $ASA \sim$.

35. Set up as for preceding theorem with medians replacing altitudes. Use $SAS \sim$ noting $\dfrac{AB}{DE} = \dfrac{\frac{1}{2}AB}{\frac{1}{2}DE}$.

36. For $\angle A$, draw angle rays AB and AC. Draw AP the angle bisector of $\angle A$. Draw perpendiculars PQ and PR from P to AB and AC respectively to form triangles $\triangle APR$ and $\triangle APQ$. To show PQ=PR use RHS$\cong$ and Theorem 24.

37. Same diagram as preceding theorem but AP is not the bisector of $\angle A$ and the perpendiculars PR and PQ are equal in length. Use Theorem 24 and $RHS \cong$.

38. Standard triangle with angle bisector CD of $\angle C$ from C to AB. Draw $AE \parallel CD$ to meet BC extended at a point E. Use Theorems 1 and 2 for angles then Theorem 17 on $\triangle ECA$ and finally Theorem 19 on $\triangle BAE$ for side ratios.

39. Use Theorem 17 and $SAS \cong$.

40. Draw line segment AB. Draw a perpendicular bisector line segment CD with D on AB. Join C to A and B. Use $SAS \cong$.

41. Draw line segment AB and plot point C above AB such that $AC = BC$. Apply Theorem 15. Draw CD with D the midpoint of AB. Use $SAS \cong$.

42. Draw $\triangle ABC$ with D and E as midpoints of AC and BC respectively. Use Theorem 20 (converse to Thales)

43. Draw $\triangle ABC$ with $\alpha = 90°$. Given D is midpoint of hypotenuse BC, plot E as midpoint of AC. Join DE and AD. Apply Theorem 42 and $SAS \cong$.

44. Apply Exercise 42 above.

45. Theorems 3 and 8.

46. Same idea as Theorem 42. A particular case of Thales Converse Theorem 20.

47. In standard triangle let AC be the longest side so $BC < AC$. Let D be a point on AC such that $CD = CB$. Join BD. Use Theorem 17 prior to labelling all the angles. Note measure of $\angle B$ is greater than each of its two components angles but by the exterior angles Theorem 16, one of these is greater than the measure of $\angle A$..

48. In standard triangle given $\beta > \alpha$, first assume $AC = BC$ and show contradiction. Second assume $AC < BC$, use Exercise 47 and show contradiction.

49. Draw standard triangle and altitude CD from C to AB. Prove $BC + AC > AB$. Note argument used in Theorem 24. Apply that argument to the two smaller right triangles.

50. Draw $\triangle ABC$ with $\angle C$ a right angle and CD the altitude from C to AB. Label all angles with α or $\beta = 90° - \alpha$. Use $AAA \sim$ on three pairs of similar triangles $\triangle ADC$, $\triangle ACB$ and $\triangle CDB$, writing the equations for the side ratios.

51. Triangle set up as for the previous exercise. Label $BC = a$, $AC = b$, $BD = c - x$ and $DA = x$ so that $AC = c$. Apply previous theorem to get $a^2 + b^2 = c^2$.

52. Easy corollary of Exercise 43.

53. Standard $\triangle ABC$ with D, E midpoints of AC and BC respectively. Label G where medians BD and AE meet and extend CG to meet AB at F. Use Exercise 44 to show $\dfrac{EG}{AG} = \dfrac{1}{2}$. Start with new $\triangle ABC$ and medians from A and C meeting at H and CH extended to I on AB. Show $\dfrac{EH}{AH} = \dfrac{1}{2}$. Conclude $H{=}G$ and medians are concurrent.

54. Build the 9 triangles in three layers, one on top, three below that and four as base. Draw equilateral $\triangle ABC$ with F marked as the centroid (where medians meet). Plot on the sides AC and BC,
$CP = PD = AD = \frac{1}{3}AC = \frac{1}{3}BC = \frac{1}{3}AB$
$CQ = QE = EB = \frac{1}{3}BC = \frac{1}{3}AC = \frac{1}{3}AB$
Top layer:
Consider top $\triangle CPQ$. Mark all segments equal to $\frac{1}{3}AB$ with $\|$. Apply Thales Theorem and conclude $\triangle CPQ$ is equilateral with all sides equal to $\frac{1}{3}AB$.
Middle layer:
Show F lies on DE. $DE \parallel AB$ by Thales. Join PF and QF. F lies on perpendicular bisector of AB and therefore of DE. So $DF = FE$. Similarly on other side. Label all $60°$ angles and sides.
Base layer:
In $\triangle CDE$, $QF \parallel AC$ by Thales Theorem so QF extended to U on AB is parallel to BC. Same with PF extended to V. Join DU and EV parallel to PV and QU respectively. Use theorems 17, 18, 19.

55. Count the triangles in previous theorem.

56. Let angle bisectors of $\angle A$ and $\angle C$ meet at $R >$ Join RB. Draw perpendiculars RP, RQ, RT to AC, BC, AB respectively. Apply$AAS \cong$ to the two triangles in $CQRP$, $ATRP$ to show QR+RT and then apply $RHS \cong$ to $BTRQ$.

57. Draw perpendicular bisectors of AB and BC to meet at G. Join G to A, B and C. Use $RHS \cong$ twice to show $AG = BG = CG$. Draw altitude GH from G to H on AC and use $RHS \cong$ to show it is the perpendicular bisector of AC.

58. Draw lines through each vertex parallel to opposite side to form a larger triangle $\triangle FED$ within which is the standard $\triangle ABC$. Label $\triangle FED$ with F opposite AC, E opposite BC and D opposite AB. Draw the altitudes AY, BX and CZ in $\triangle ABC$. Identify parallelograms to show $BD = BE$, $AF = AD$ and $FC = CS$. Conclude the altitudes of $\triangle ABC$ are the perpendicular bisectors of $\triangle FED$ which are concurrent by Theorem 57.

59. Draw $\triangle ABC$ with (for clarity) $\angle A > \angle B$ and Q the midpoint of AB. Let D and G be the circumcenter (where perpendicular bisectors meet) and centroid. Join CQ and position G such that $CG = 2GQ$. Project DG to a point H such that $DG = \frac{1}{2}GH$. Project CH to meet AB at P. In $\triangle CGH$ and $\triangle DQG$,

$$\frac{DG}{GH} = \frac{GQ}{CG} = \frac{1}{2}$$

so by Thales Theorem $DQ \parallel CH$. But $DQ \perp AB$ so CP is an altitude. Similarly BH extended to AC is perpendicular to AC and so H is the orthocenter.

60. Position D anywhere in standard triangle $\triangle ABC$. Extend BD to a point E on AC and AD to a point F on BC. Draw a line through C parallel to AB and extend BE and AF to meet this line at R and S respectively. Project CD to meet AB at G. Start with the identity $\dfrac{CS}{CR} \times \dfrac{AB}{CS} \times \dfrac{CR}{AB} = 1$ and make substitutions into it, one from $\triangle AFB \sim \triangle CFS$, one from $\triangle AEB \sim \triangle RCE$ and the third from a combination of the equations of $\triangle RCD \sim \triangle DGB$ and $\triangle AGD \sim \triangle CDS$.

61. (Napoleon) Draw any triangle and erect equilateral triangles on the three sides. Locate their centroids and label them A, B, C. Rotate BC clockwise by $120°$ about C and label the image of B as D. Rotate BA counterclockwise by $120°$ about A and again the image of B is D. Then $ABCD$ is a kite (two unequal isosceles triangles with a common base), so AC is angle bisector of both $\angle BAD$ and $\angle BCD$ by $SAS \cong$ on $\triangle ABC$ and $\triangle ADC$ and all the angles of $\triangle ABC$ measure $60°$.

62. (Viviani) Draw equilateral triangle $\triangle ABC$ and position a point P inside it. Draw perpendiculars from P to the three sides. The area of $\triangle ABC$ is equal to the sum of the areas of the triangles formed by joining P to A, B, and C. Use $Area = \frac{1}{2} base \times height$. Note $\triangle ABC$ is an equilateral triangle.

63. (Pompeiu) Draw equilateral triangle $\triangle ABC$ overlaid by a congruent clone. Position (for the sake of a simple diagram) P almost directly above B and slightly higher than C. Rotate the clone clockwise about B by $60°$ so that A maps to C and P maps to Q. Note $BP = BQ$ and $\angle PBQ = 60°$ so $\triangle PBQ$ is equilateral. Conclude $\triangle PCQ$ has the sides equal to PA, PB and PC.

64. (Apolloniu) On a set of axes plot $A(-a, 0), B(a, 0), C(b, c)$ and $D(0, 0)$. Use the distance formula to show $CA^2 + CB^2 = 2(AD^2 + CD^2)$

65. (Morley's Theorem with proof by HD Grossman) Let the triangle have base BC and angles $3\alpha, 3\beta, 3\gamma$. The lines m and n will meet at A. Let BD and CD be angle trisectors as well as BF and CE. The positions of E and F are determined by making $m\angle CDE = 60° + \beta$ and $m\angle BDF = 60° + \gamma$. Show $m\angle FDE = 60°$ and using $3\alpha + 3\beta + 3\gamma = 180°$ that $m\angle BFD = m\angle CED = 60° + \alpha$. Accept $DE = DF$ since D is equidistant from BF and CE. (Leave this sub-proof to later.) So $\triangle DEF$ is an equilateral triangle.

Extend BD into $\triangle CDE$ to a point K and likewise CD to H. Find $\angle FDH$ and $\angle EDK$ in terms of β and γ using Theorem 16. Position H and K so that FHD and DKE are isosceles triangles and extend HF to become the (adjusted) line r and KE to become (adjusted) line s. Find $m\angle HFB$ in terms of α and β. Show the angle between lines m and r measures α and that this is also the angle between lines s and n. Confirm the angle between m and n measures 3α. Confirm the lines m, r, s, n all converge to a point (labelled A). You need to prove the lines m, n, s, and r converge to the point A. Note the

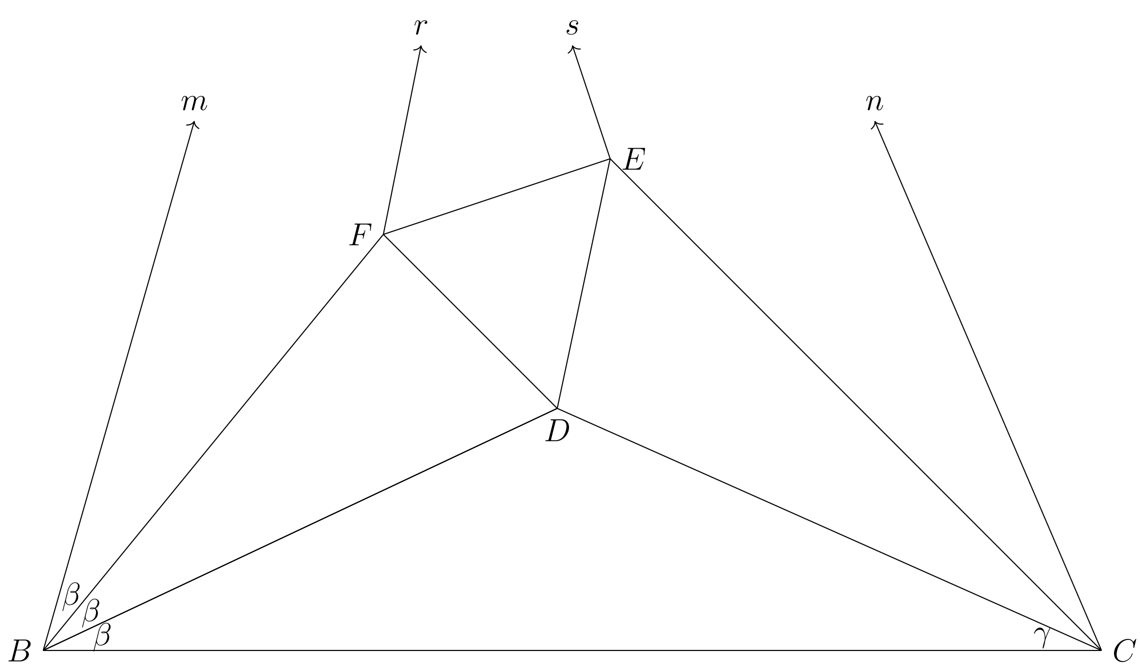


Figure 59: Morley's Theorem

line KF joins the vertices of two isosceles triangles and therefore bisects $\angle K$. Then in $\triangle mBKs$, the bisector of $\angle ms$ passes through F and being paralle to r must coincide with it. Similarly on the other side.

Chapter 12

Parallelograms

The next set of plane figures to be investigated have four sides and are called quadrilaterals. There are five sub-categories. You are invited to prove the main theorems.

12.1 Definitions

Definition 23. *A quadrilateral is a plane figure with four sides, four interior angles and four vertices.*

Definition 24. *A parallelogram is a quadrilateral with two pairs of parallel sides.*

Definition 25. *A trapezoid is a quadrilateral with one pair of parallel sides.*

Definition 26. *A rectangle is a parallelogram with four right angles as its interior angles (actually, one would suffice.)*

Definition 27. *A rhombus is a parallelogram with four congruent sides.*

Definition 28. *A square is a rectangle with four congruent sides or a rhombus with four right angles.*

12.2 Exercises or Theorems to be proved

1. Consecutive pairs of sides of a parallelogram are supplementary. (Consecutive means pairs of angles as you go clockwise or counter-clockwise around the interior angles.)

2. Opposite angles of a parallelogram are congruent.

3. The opposite sides of a parallelogram are congruent to one another.

4. Either diagonal separates a parallelogram into two congruent triangles.

5. The diagonals of a parallelogram bisect each other.

6. The diagonals of a parallelogram bisect each other.
7. The diagonals of a rectangle are congruent.
8. The diagonals of a rhombus bisect the four interior angles of the rhombus.
9. The diagonals of a rhombus are perpendicular to one another.
10. If a quadrilateral has one pair of sides that are parallel and congruent then it is a parallelogram.
11. A quadrilateral is a parallelogram if opposite sides are congruent.
12. A quadrilateral is a parallelogram if opposite angles are congruent.
13. A quadrilateral is a parallelogram if its diagonals bisect one another.
14. (The median of a trapeziod is the line joining the midpoints of the non-parallel sides.) The median of a trapeziod is parallel to the bases and has a length equal to one-half the sum of the lengths of the bases.
15. If the opposite sides of a quadrilateral are congruent then it is a paralleogram.
16. If one angle of a parallelogram is 90° then it is a rectangle.
17. If the diagonals of a parallelogram are equal then it is a rectangle.
18. If the diagonals of a parallelogram are perpendicular then it is a rhombus.
19. If one diagonal of a parallelogram bisects a pair of opposite angles then the parallelogram is a rhombus.
20. If one pair of consecutive sides of a parallelogram are congruent then the parallelogram is a rhombus.

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