PRECALCULUS

VOLUME 2

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Chapter 5

Trigonometric Functions

5.1 Angle Measurement

The standard way to draw an angle on the Cartesian plane is to draw two rays or half-lines out from the origin.

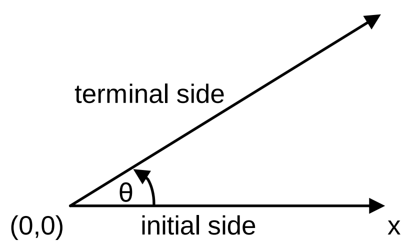

Figure 5.1 Positive angles

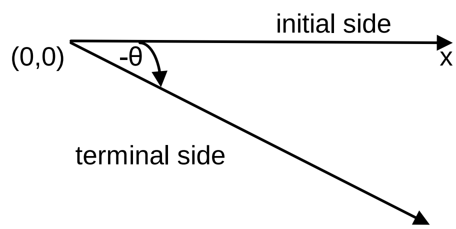

Figure 5.2 Negative angles

One of the rays is the positive $x-axis$. We call this the initial side of the angle. The other is called the terminal side. The terminal side starts also on the positive $x-axis$ and rotates counterclockwise to form positive angles (Figure 5.1) and clockwise to form negative angles (Figure 5.2).

5.2 Degrees

Definition. If the terminal side rotates counterclockwise to finish on the positive $y-axis$, we measure the angle as 90 degrees or 90^0. See the first diagram below.

This leads to the following examples of angles measured in degrees.

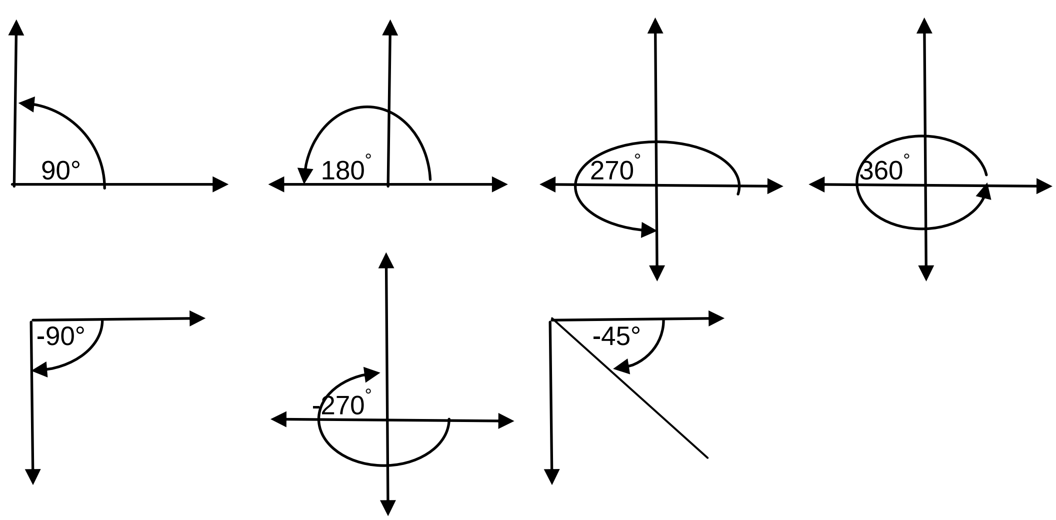

Figure 5.3

Angles can be formed by more than one rotation. Hence we have:

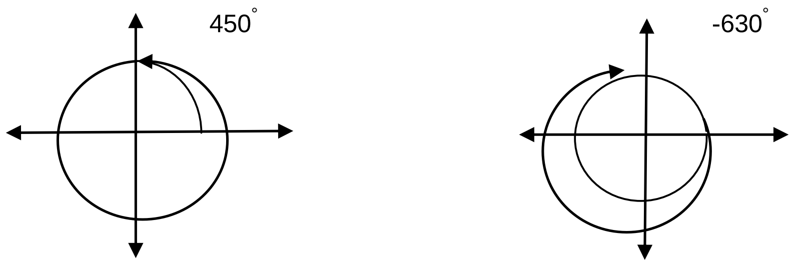

Figure 5.4

5.3 Radians

If we draw a circle of radius r centered at the origin, its circumference is $2\pi r$. We use the symbol s for the length of an arc of the circle.

Definition. An angle of measure **one radian** is the angle subtended[1] at the origin of a circle of radius r by an arc of the circle of length $s = r$. See Figure 5.5.

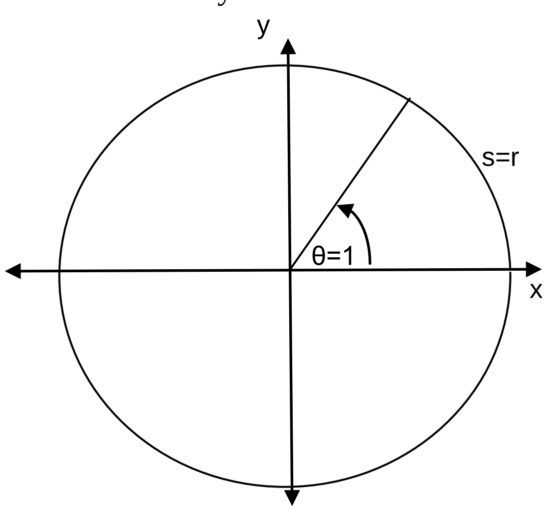

Figure 5.5 Angle of 1 radian

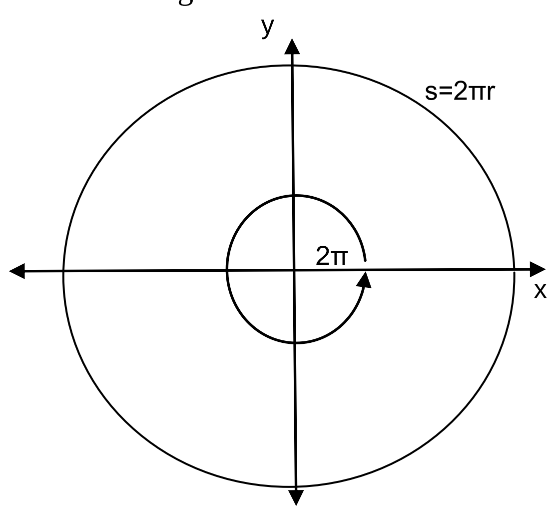

Figure 5.6 Angle of 2π radians

Algebraically, this means $s = r\theta \Leftrightarrow \theta = \dfrac{s}{r}$ since if $s = 2r$, $\theta = 2$ radians, if $s = 3r$, $\theta = 3$ radians and so on.

If we take one complete revolution, then the arc length is simply the circumference of the circle, that is, $s = 2\pi r$. Then $\theta = \dfrac{s}{r} = \dfrac{2\pi r}{r} = 2\pi$ radians or the angle subtended by the whole circle at the center is 2π radians. See Figure 5.6.

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[1] "Subtended " means formed by the rays drawn from the ends of the arc to the center.

5.4 Radians and Degrees

Since the angle subtended by the whole circle at the center is also 360^0, it follows that 2π radians $= 360^0$, hence,

$$1 \text{ radian } = \frac{180^0}{\pi}$$

$$1^0 = \frac{\pi}{180} \text{ radians.}$$

The common angles are related as follows. Note, when we write an angle in radians, we normally omit the word "radians", that is $\pi = \pi$ radians.

Degrees	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

Example 1

Convert $(a)\, 135^0$ $(b)\, 540^0$ $(c)\, -210^0$ to radians

<u>Solution</u>

(a) $135^0 = 135\frac{\pi}{180} = \frac{3\pi}{4}$

(b) $540^0 = 540\frac{\pi}{180} = 3\pi$

(c) $-210^0 = -210\frac{\pi}{180} = -\frac{7\pi}{6}$

Example 2

Convert $(a) -\frac{\pi}{2}$ $(b)\frac{7\pi}{3}$ $(c)\, 2$ to degrees.

Solution

$(a) -\frac{\pi}{2} = -\frac{\pi}{2} \bullet \frac{180^0}{\pi} = -90^0$

$(b)\frac{7\pi}{3} = \frac{7\pi}{3} \bullet \frac{180^0}{\pi} = 420^0$

$(c)\, 2 = 2 \bullet \frac{180^0}{\pi} \approx 114.59^0$

5.5 Arc Length

From $\theta = \frac{s}{r}$, the formula for the length of an arc that subtends an angle of θ radians at the center of a circle of radius r is: $s = r\theta$. See Figure 5.7.

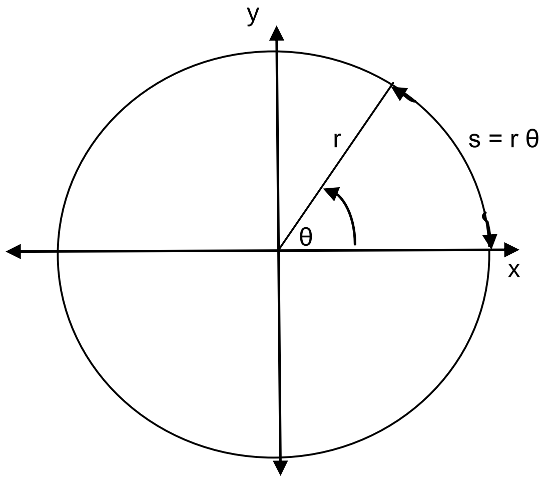


Figure 5.7 Arc Length

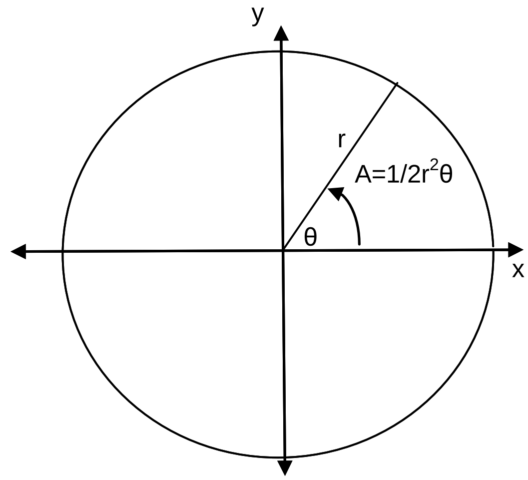


Figure 5.8 Area of Sector

Example 3

A circle has radius 3. What is the length of an arc that subtends an angle of 330^0 at the origin?

Solution

$$330^0 = 330 \bullet \frac{\pi}{180} = \frac{11\pi}{6};$$

$$s = r\theta = 3 \bullet \frac{11\pi}{6} = \frac{11\pi}{2} \approx 17.31$$

5.6 Area of a sector of a circle

A sector of a circle is the region bounded by an arc that subtends an angle θ at the center and the two radii from the center to the ends of the arc. Since the area of the circle subtended by the angle 2π is $A = \pi r^2 = \frac{1}{2} r^2 2\pi$, then proportionally, the area of the sector subtend by the angle θ is $A = \frac{1}{2} r^2 \theta$. See Figure 5.8.

Warning: Some calculators need radians as the input, others use degrees.

Example 4

What is the area of the sector of the circle that subtends an angle of 20^0 at the origin if the radius of the circle is 5?

Solution

$$20^0 = 20 \bullet \frac{\pi}{180} = \frac{\pi}{9}$$

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} \bullet 25 \bullet \frac{\pi}{9} = \frac{25\pi}{18} \approx 4.4$$

5.7 Two Special Triangles

There are two special triangles that will prove very useful in our study of trigonometric functions.

5.7.1 Isosceles Right Triangle or $45^0 - 45^0 - 90^0$ triangle.

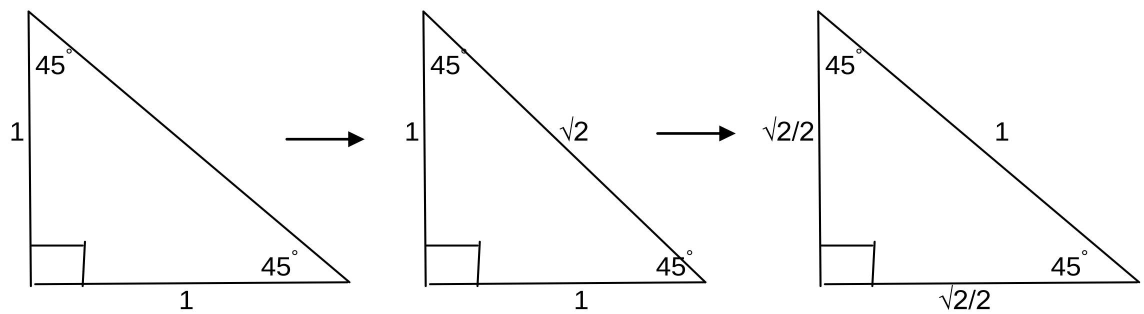


Figure 5.9

By the Pythagorean Theorem, (in a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides), if two equal sides of a right triangle are both 1 unit, then the hypotenuse is $\sqrt{2}$. In the final step we simply multiplied each side by the same factor, namely $\frac{\sqrt{2}}{2}$.

5.7.2 The $30^0 - 60^0$ Right Triangle

If we add on a mirror image of a $30^0 - 60^0 - 90^0$ triangle, we get an equilateral triangle with all angles equal to 60^0 - see Figure 5.10. Hence all the sides are also equal, say to 2. Then we have Figure 5.11, again by using the Pythagorean Theorem, and proportionally, Figure 5.12.

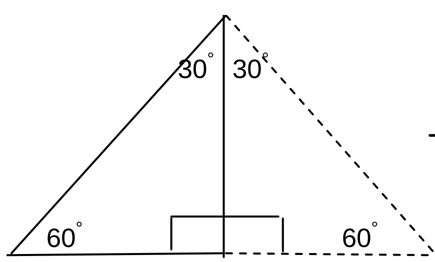


Figure 5.10

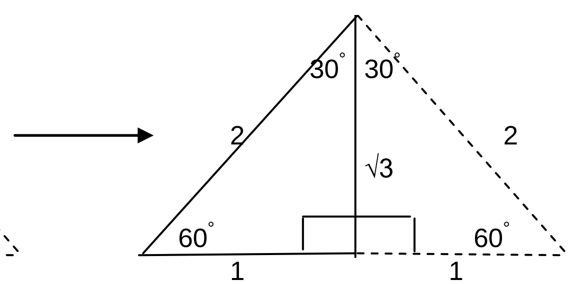


Figure 5.11

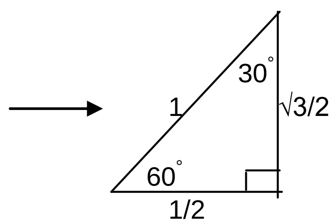


Figure 5.12

5.8 The Trigonometric Functions

5.8.1 The Unit Circle

The unit circle is centered at the origin and has radius 1. It has equation $x^2 + y^2 = 1$.

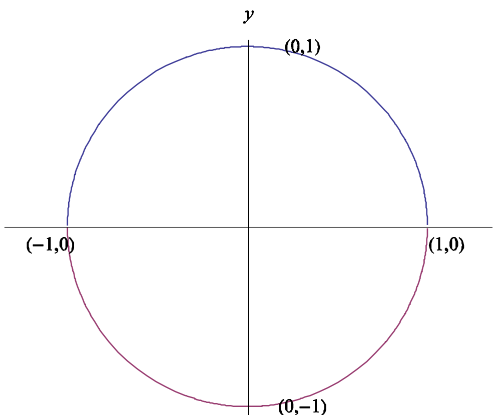


Figure 5.13 The Unit Circle

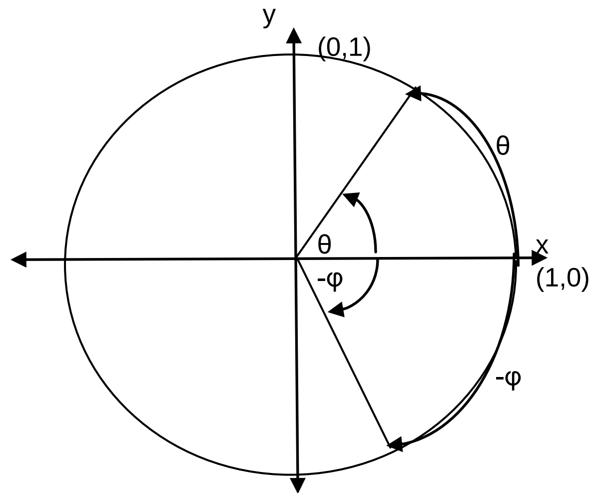


Figure 5.14

Now, an angle of one radian is defined to be the angle subtended at the origin by an arc of length 1. If therefore, we wrap a number line around the unit circle so that its origin or 0 is at $(1,0)$, its positive part is wrapped counterclockwise and its negative part is wrapped clockwise, then any real numbers on the number line from $-\infty$ to $+\infty$, say, θ and $-\phi$, correspond to a subtended angle with the same radian measures of θ or $-\phi$. Figure 5.14.

This remains true if we wrap the number line multiple times around the unit circle. For example, the number $19.37 = 6\pi + 0.5204$ on the number line will end up on the circumference of the unit circle and be in the first quadrant. As we wrap the number line around three times and a bit more, the terminal side of the angle goes three times and a bit more around the origin and ends up forming an angle of $6\pi + 0.5204 = 19.37$ radians. Note, it overlays the angle $0.5204 = 29.82^0$ in the first quadrant as shown in Figure 5.15.

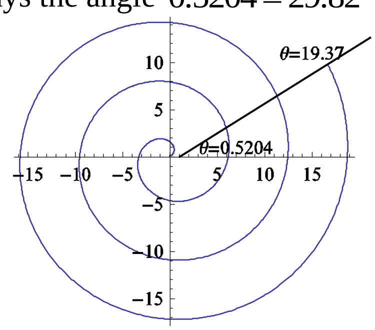


Figure 5.15 The angle 19.27 overlaying the angle 0.5204

5.8.2 Definitions of Trigonometric Functions

Let θ be a real number and $P(x, y)$ be the point on the (wrapped) unit circle corresponding to a subtended angle θ.

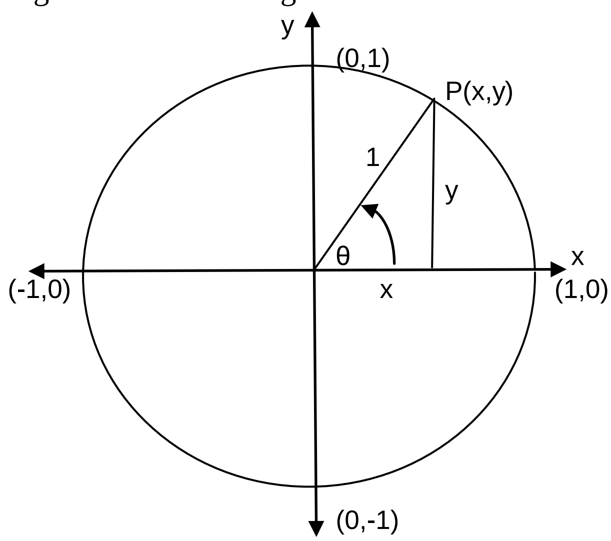


Figure 5.16

We define the trigonometric functions sine, cosine and tangent by,

Definitions

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \ (x \neq 0)$$

5.8.3 Pythagorean Trigonometric Identity

Since the equation of the unit circle is $x^2 + y^2 = 1$, it follows that,

$$\sin^2\theta + \cos^2\theta = 1$$

This is called the Pythagorean trigonometric identity.

5.8.4 Domain and Range

The domain of the sine and cosine functions is all real numbers since we wrap the whole real number line around the unit circle.

The range of the sine and cosine functions is $[-1, +1]$ since x, y are values of the coordinates of points on the unit circle where $-1 \leq x, y \leq 1$.

The tangent function $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is not defined if

$\cos\theta = x = 0 \Leftrightarrow \theta = \dfrac{\pi}{2} + 2n\pi \text{ or } \dfrac{3\pi}{2} + 2n\pi = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}.$[2]

We write the domain of $y = \tan\theta$ as

$\left\{\theta : \theta \neq \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\right\} \Leftrightarrow \left\{\theta^0 : \theta^0 \neq 90^0 + n \bullet 180^0, n \in \mathbb{Z}\right\}$. Its range is $(-\infty, +\infty)$.

5.8.5 Periodicity of the Sine and Cosine Functions

If we wrap the positive half of the real number line around the unit circle with the origin 0 of the number line placed on the point $(1,0)$ on the unit circle, then the origin gets overlaid successively by $2\pi, 4\pi, 6\pi, ..., 2n\pi$ where n is any positive integer. If we wrap the other negative half of the real number line around the unit circle in a clockwise direction, then the origin is overlaid successively by $-2\pi, -4\pi, -6\pi, ..., 2n\pi$ where n is any negative integer. In other words, the origin is overlaid by $2n\pi, n \in \mathbb{Z}$.

This occurs at every point on the circumference of the unit circle. The general point θ is overlaid by $\theta + 2n\pi, n \in \mathbb{Z}$. Refer back to Figure 5.15 as an example.

It follows that if $P(x, y)$ is any point on the unit circle and $\theta + 2n\pi, n \in \mathbb{Z}$ is the subtended angle, then,

$$\sin(\theta + 2n\pi) = \sin\theta$$
$$\cos(\theta + 2n\pi) = \cos\theta$$

We use this property when you want to find values of these functions outside $[0, 2\pi]$. For example, from Figure 5.15, $\sin 19.27 = \sin(6\pi + 0.5204) = \sin 0.5204$. We say sine and cosine are periodic functions with period 2π. More generally,

Definition

*A function $y = f(x)$ is called **periodic** if there exists a number p such that*

$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest such number p is called the period of the function.

In the case of the sine function, for example,

$$\sin(\theta + 2\pi) = \sin\theta,$$
$$\sin(\theta + 4\pi) = \sin(\theta + 2\pi) = \sin\theta$$

[2] You should convince yourself that $\dfrac{\pi}{2} + 2n\pi \text{ or } \dfrac{3\pi}{2} + 2n\pi = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$ by writing out the first few terms of each of the three sequences.

5.8.6 Evaluating Trigonometric Functions at common values of the variable.

With hypotenuse equal to 1 for both, if we superimpose the $45^0-45^0-90^0$ triangle and the $30^0-60^0-90^0$ triangle on the unit circle and rotate them through the four quadrants, we obtain the two diagrams shown in Figure 5.17.

We can use these figures and periodicity to evaluate trigonometric functions at $\theta+2\pi n, n\in\mathbb{Z}$ for $\theta=0^0,30^0,45^0,60^0,$

$90^0,180^0,270^0,360^0$

or,

$$\theta=0,\pi/6,\pi/4,\pi/3,$$
$$\pi/2,\pi,3\pi/2,2\pi$$

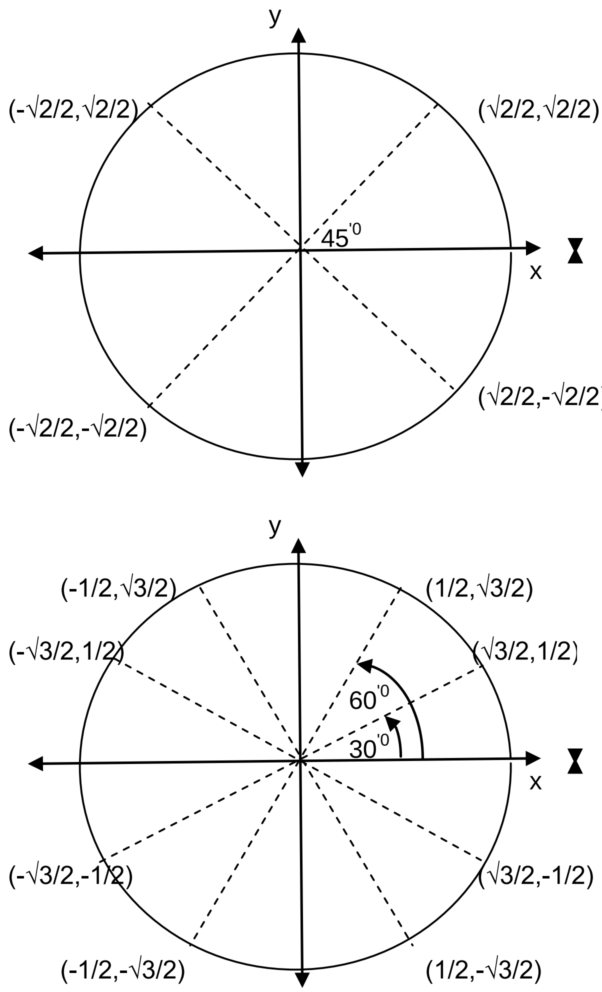


Figure 5.17

You should memorize this table. We use these values all the time.

θ^0	0^0	30^0	45^0	60^0	90^0	180^0	270^0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.	0	undef.

In addition, as we saw earlier, wrapping $\theta + 2n\pi$ around the unit circle results in angles overlaying θ, so we have,

$$\sin(\theta + 2n\pi) = \sin\theta \quad or \quad \sin(\theta^0 + n360^0) = \sin\theta^0$$
$$\cos(\theta + 2n\pi) = \cos\theta \quad or \quad \cos(\theta^0 + n360^0) = \cos\theta^0$$
$$\tan(\theta + 2n\pi) = \tan\theta \quad or \quad \tan(\theta^0 + n360^0) = \tan\theta^0$$

Example 5
Evaluate the sine and cosine functions at $\pi/6, \pi, 3\pi/2, 2\pi$
Solution Read the answers off from Figure 5.17 or the Table below it.
$\sin\pi/6 = 1/2,\ \sin\pi = 0,\ \sin 3\pi/2 = -1,\ \sin 2\pi = 0$
$\cos\pi/6 = \sqrt{3}/2,\ \cos\pi = -1,\ \cos 3\pi/2 = 0,\ \cos 2\pi = 1$

Example 6
Evaluate $\sin 13\pi/6,\ \cos(-7\pi/2)$
Solution
$\sin 13\pi/6 = \sin(2\pi + \pi/6) = \sin\pi/6 = 1/2$
$\cos(-7\pi/2) = \cos(-4\pi + \pi/2) = \cos(\pi/2) = 0$

5.8.7 Values of Trigonometric Functions of Any Real Number

Definition Reference Angle

Consider any angle in the standard position. We define its **reference angle** to be the acute angle θ' formed by its terminal side and the $x-axis$.

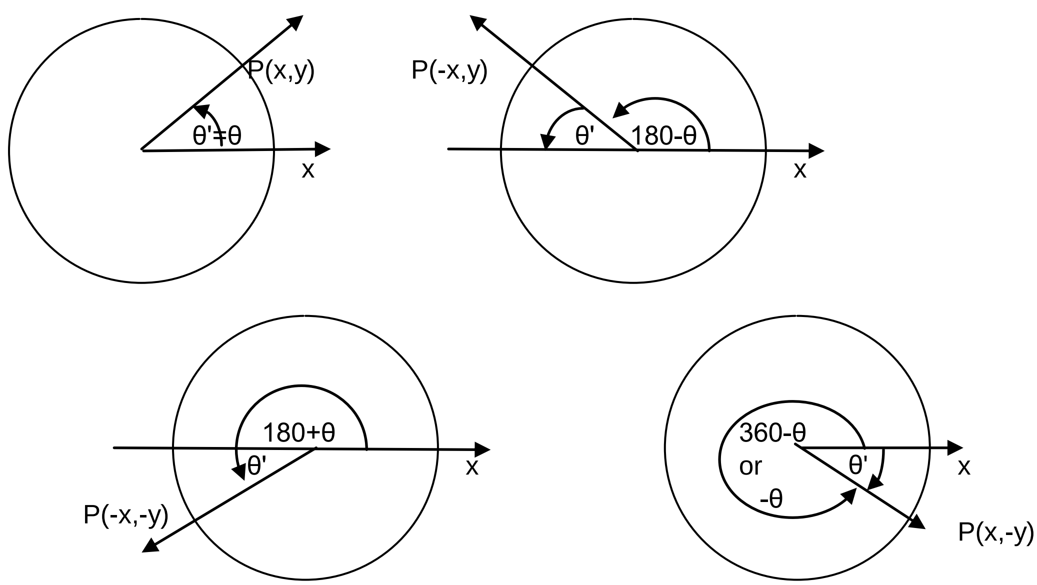


Figure 5.18

In the above figures, we start with an acute angle θ in the first quadrant of the unit circle. If its terminal side meets the unit circle at $P(x, y)$, by definition,

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{y}{x}$$

In the second figure we consider the angle $180^0 - \theta$ which will be in the second quadrant. Its reference angle θ' is equal to θ. Therefore its terminal side meets the unit circle at $(-x, y)$. Hence, by definition,

$$\sin(180^0 - \theta) = y = \sin\theta$$

$$\cos(180^0 - \theta) = -x = -\cos\theta$$

$$\sin(180^0 - \theta) = \frac{y}{-x} = -\tan\theta$$

In the third figure we consider the angle $180^0 + \theta$ which will be in the third quadrant. Its reference angle θ' is again equal to θ. Therefore its terminal side meets the unit circle at $(-x, -y)$. Hence, by definition,

$$\sin(180^0 + \theta) = -y = -\sin\theta$$

$$\cos(180^0 + \theta) = -x = -\cos\theta$$

$$\tan(180^0 + \theta) = \frac{-y}{-x} = \tan\theta$$

In the fourth figure we consider the angles $360^0 - \theta$ or $-\theta$ which will be in the fourth quadrant. Their reference angle θ' is also equal to θ. Therefore their terminal side meets the unit circle at $(x, -y)$. Hence, by definition,

$$\sin(360^0 - \theta) = -y = -\sin\theta \qquad \sin(-\theta) = -y = -\sin\theta$$

$$\cos(360^0 - \theta) = x = \cos\theta \qquad \cos(-\theta) = x = \cos\theta$$

$$\tan(360^0 - \theta) = \frac{-y}{x} = -\tan\theta \qquad \tan(-\theta) = \frac{-y}{x} = -\tan\theta$$

We summarize this information in Figure 5.19.

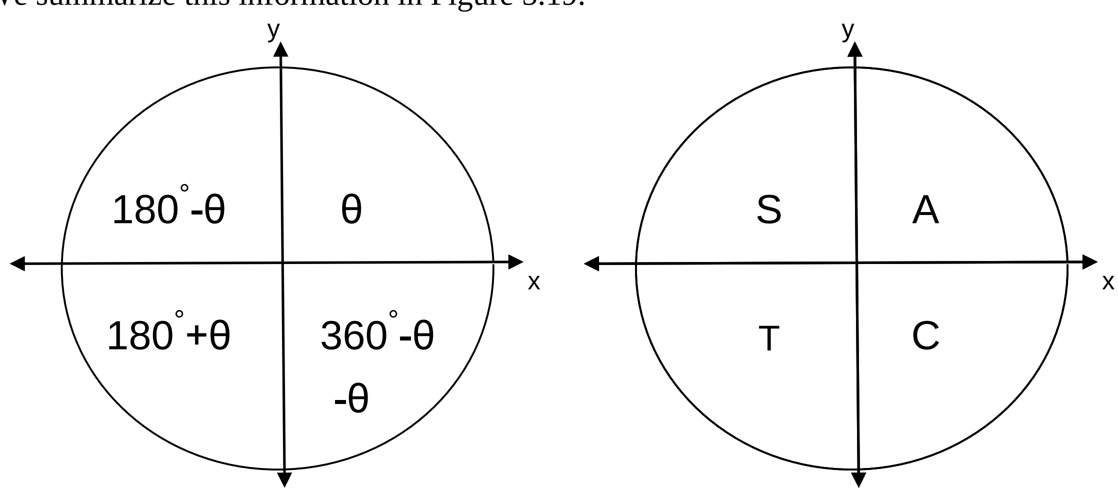


Figure 5.19

Generations of students remember the second diagram as "All Students Take Calculus" or "All Stations To Central". It means, for angles in the first quadrant, All trig functions take positive values; for angles in the second only Sine is positive; for angles in the third only Tangent is positive and for angles in the fourth quadrant only Cosine is positive.

For radian measure, the formulas are,

$$
\begin{array}{llll}
\sin(\pi-\theta)=\sin\theta & \sin(\pi+\theta)=-\sin\theta & \sin(2\pi-\theta)=-\sin\theta & \sin(-\theta)=-\sin\theta \\
\cos(\pi-\theta)=-\cos\theta & \cos(\pi+\theta)=-\cos\theta & \cos(2\pi-\theta)=\cos\theta & \cos(-\theta)=\cos\theta \\
tan(\pi-\theta)=-\tan\theta & tan(\pi+\theta)=\tan\theta & tan(2\pi-\theta)=-\tan\theta & tan(-\theta)=-\tan\theta
\end{array}
$$

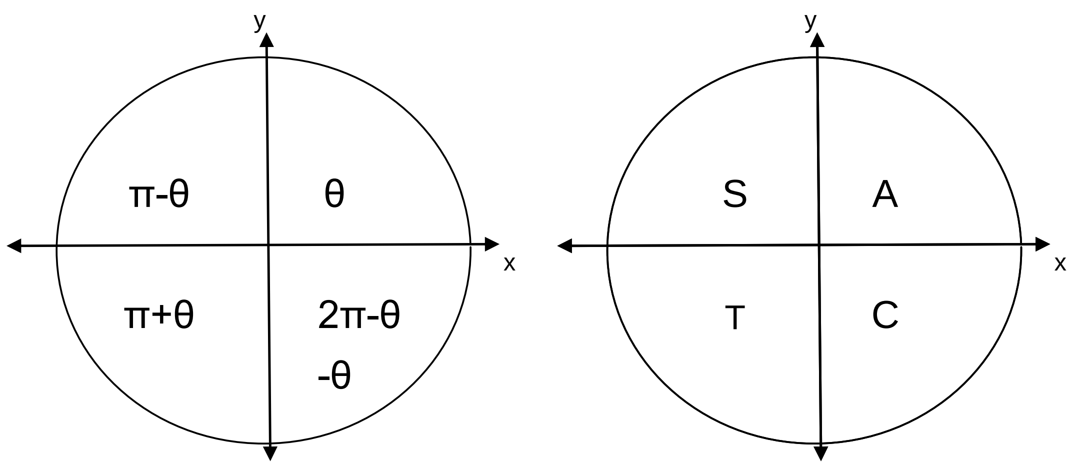


Figure 5.20

To evaluate the trigonometric value of any angle, the method is to express the given angle in the form $\theta, 180^0-\theta, 180^0+\theta$ *or* $360^0-\theta$ *or* $-\theta$, (or $\theta, \pi-\theta$, etc). where θ is an acute angle. You may need to subtract or add multiples of 360^0 (*or* 2π) first. Then the trigonometric value of the angle is the same as the trigonometric value of the acute angle θ, with the sign given by ASTC.

Example 7
Evaluate $\sin 210^0$
Solution We write the angle in the one of the forms in the first diagram of Figure 5.19. In this case, $\sin 210^0=\sin(180^0+30^0)$ placing the angle in the third quadrant where only tan is positive. Hence $\sin 210^0=\sin(180^0+30^0)=-\sin 30^0=-\frac{1}{2}$

Example 8
Evaluate $\cos 135^0$

Solution $\cos 135^0 = \cos(180^0 - 45^0) = -\cos 45^0 = -\frac{\sqrt{2}}{2}$

Example 9

Evaluate $\tan\frac{5\pi}{6}$

Solution

Working in radians rather than degrees and using Figure 5.20,

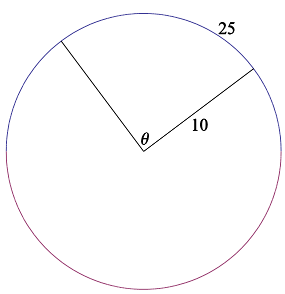
$$\tan\frac{5\pi}{6} = \tan(\pi - \frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Exercises 5A

1. Determine the quadrant in which (a) -200^0 and (b) -17^0 lie.
2. Sketch (a) 180^0 and (b) 135^0 in the standard position.
3. Change to radian measure: (a) 315^0, (b) 120^0
4. Change to degree measure: (a) $-\frac{5\pi}{6}$, (b) $\frac{\pi}{12}$
5. Change to degree measure: (a) $-\frac{5\pi}{3}$, (b) $\frac{14\pi}{15}$
6. Convert from degrees to radians, rounding to 3 decimal places: (a) 69.3^0, (b) -101.1^0
7. Find the length of the arc of a circle of radius 9 feet and central angle 60^0.

In each of 8 and 9, find the angle θ in radians.

8.



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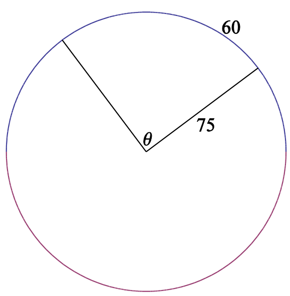
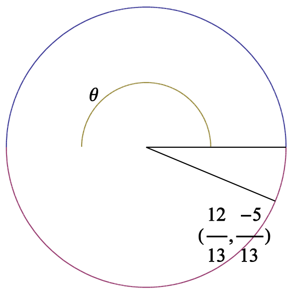


10. A sprinkler on a golf course is set to spray water over a distance of 15 meters and to rotate through an angle of 140^0. Draw a diagram that shows the region that can be irrigated with the sprinkler. Find the area of the region.
11. Determine the exact values of the six trigonometric functions of the real number θ.



In 12 and 13 find the point on the unit circle that corresponds to the real number t.

12. $\theta = \pi$
13. $\theta = \dfrac{3\pi}{4}$

In 14, 15, 16 and 17 find the sine, cosine and tangent of the real number.

14. $\theta = \dfrac{\pi}{3}$
15. $\theta = -\dfrac{\pi}{4}$
16. $\theta = -\dfrac{4\pi}{3}$
17. $\theta = \dfrac{5\pi}{3}$

In 18, 19, 20 and 21, evaluate, if possible, the six trigonometric functions of the real number.

18. $\theta = \dfrac{5\pi}{6}$
19. $\theta = \dfrac{7\pi}{4}$
20. $\theta = \dfrac{3\pi}{2}$
21. $\theta = -\pi$

22. Evaluate $\cos 3\pi$ using periodicity.
23. Evaluate $\sin \dfrac{9\pi}{4}$ using periodicity.
24. If $\sin(-\theta) = \dfrac{3}{8}$, find (a) $\sin\theta$ (b) $\csc\theta$

25. If $\tan(-\theta) = 2$ find $\cot\theta$

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5.8.8 Solving trigonometric equations

To solve equations, we need to reverse our thinking.

Example 10

Solve $\sin\theta = \frac{1}{2},\ \theta \in \mathbb{R}$

Solution Since sine is positive in both the first and second quadrants, both $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin(\pi - \frac{\pi}{6}) = \sin\frac{5\pi}{6} = \frac{1}{2}$. This gives solutions in the domain $[0, 2\pi]$ as $\frac{\pi}{6}, \frac{5\pi}{6}$. Then we use $\sin(\theta + 2\pi n) = \sin\theta,\ n \in \mathbb{Z}$ to obtain the general solutions $\frac{\pi}{6} + 2\pi n,\ \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z}$.

Example 11

Solve $\cos\theta = -\frac{1}{2}, \theta \in \mathbb{R}$

Solution Cosine is negative in both the second and third quadrants. The first quadrant solution to $\cos\theta = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$.

This gives solutions in the domain $[0, 2\pi]$ as $\pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$ or $\frac{2\pi}{3}, \frac{4\pi}{3}$. The general solutions are $\frac{2\pi}{3} + 2n\pi,\ \frac{4\pi}{3} + 2n\pi,\ n \in \mathbb{Z}$.

Example 12

Solve $\tan\theta = 1$ for $\theta \in [0, 2\pi]$

Solution The tangent function is positive in the first and third quadrants where angles have the format $\theta, \pi + \theta$

Since $\tan\frac{\pi}{4} = 1$, the solutions are $\frac{\pi}{4}, \pi + \frac{\pi}{4} = \frac{\pi}{4}, \frac{5\pi}{4}$.

Example 13

Find all solutions for $2\sin^2\theta + \sin\theta - 1 = 0$

Solution

$2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$

$\Rightarrow \sin\theta = 1/2,\ \sin\theta = -1$

$\Rightarrow \theta = \pi/6 + 2n\pi, 5\pi/6 + 2n\pi, 3\pi/2 + 2n\pi, n \in \mathbb{Z}$

Exercises 5B

In 1 to 14 find all solutions of the given equation.

1. Solve $\sin\theta = \frac{\sqrt{3}}{2}$
2. Solve $\tan\theta = -\sqrt{3}$
3. Solve $\cos\theta = -\frac{\sqrt{3}}{2}$
4. Solve $\sqrt{2}\sin\theta + 1 = 0$
5. Solve $\sqrt{2}\cos\theta - 1 = 0$
6. Solve $\tan^2\theta - 1 = 0$
7. Solve $4\cos^2\theta - 4\cos\theta + 1 = 0$
8. Solve $2\sin^2\theta - \sin\theta - 1 = 0$
9. Solve $2\sin x + 1 = 0$
10. Solve $\tan x + \sqrt{3} = 0$
11. Solve $(3\tan^2 x - 1)(\tan^2 x - 3) = 0$
12. Solve $\tan^2 3x = 3$
13. Solve $\sec^2 x - 1 = 0$
14. Solve $2\sin^2 x + 3\sin x + 1 = 0$

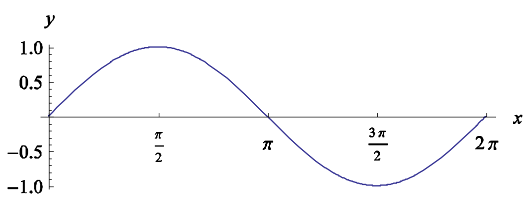
Chapter 6

Graphs of the Trigonometric Functions

6.1 Graphs of sine and cosine functions

We will use x as the independent variable rather than θ. As we saw, the domain of the sine and cosine functions is $(-\infty,\infty)$, the range is [-1,1] and the functions are periodic with period 2π. Moving counterclockwise around the unit circle from $(1,0)$, the graph of $y=\sin x$ starts at $y=0$ and $x=0$, increases to $y=1$ at $x=\pi/2$, decreases back to $y=0$ at $x=\pi$, then decreases further to $y=-1$ at $x=3\pi/2$, before returning to $y=0$ at $x=2\pi$. See Figure 6.1. Similarly, for the graph of $y=\cos x$ which takes the values $(0,1), (\pi/2,0), (-\pi,-1), (3\pi/2,0), (2\pi,1)$. See Figure 6.2.

We call the section of each graph from $0\le x\le 2\pi$ the fundamental cycle of the graph. This cycle is repeated infinitely in both the positive and negative directions. (Figure 6.3).



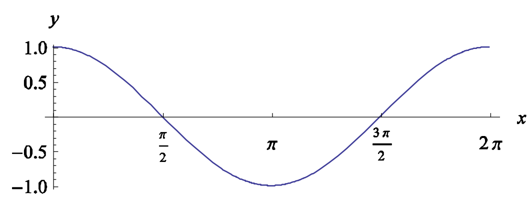


Figure 6.1 Fundamental cycle of $y=\sin x$ Figure 6.2 Fundamental cycle of $y=\cos x$

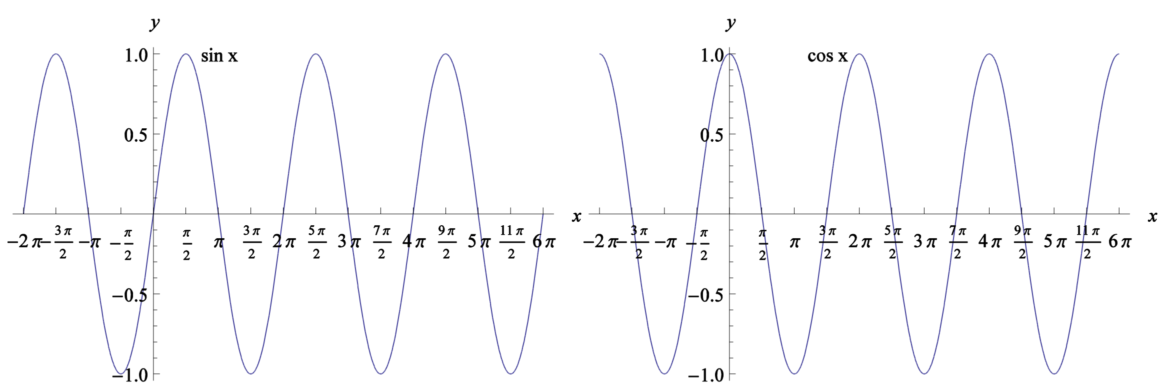


Figure 6.3 Extended graphs of $y=\sin x$ and $y=\cos x$

Notes: (1) $y=\sin x$ intercepts the $x-axis$ at $n\pi,\ n\in\mathbb{Z}$

(2) $y=\sin x$ has maxima at $\dfrac{\pi}{2}\pm 2\pi n,\ n\in\mathbb{Z}$

(3) $y=\sin x$ has minima at $\dfrac{3\pi}{2}\pm 2\pi n,\ n\in\mathbb{Z}$

(5) $y = \cos x$ intercepts the $x-axis$ at $\frac{n\pi}{2}$, n an odd integer

(6) $y = \cos x$ has maxima at $0 \pm 2\pi n,\ n \in \mathbb{Z}$

(7) $y = \cos x$ has minima at $\pi \pm 2\pi n,\ n \in \mathbb{Z}$

Definitions

Amplitude: The amplitude of the sine and cosine graphs is 1. In general, it is half the distance between the maximum and minimum values of y.

Period: The period is the length of the fundamental cycle. It is 2π for both $y = \sin x$ and $y = \cos x$.

6.2 Transformations of the sine and cosine graphs

We can use our transformation rules for the translation, reflection, stretching and shrinking of the basic sine and cosine graphs. Let's recall the summary rules.

Transformation Summary Rule

The graph of $y - k = af(b(x-h))$ is obtained from the basic graph of $y = f(x)$ as follows.

(a) If $a < 0$, reflect the basic graph in the $x-axis$. If $|a| < 1$, the basic graph is shrunk vertically by that factor, if $|a| > 1$ the basic graph is stretched vertically by that factor. In the case of sine and cosine functions, the amplitude will therefore be $|a|$.

(b) If $b < 0$, reflect the graph in the $y-axis$. If $|b| > 1$, the basic graph is shrunk horizontally by a factor of b. If $|b| < 1$, the basic graph is stretched horizontally by a factor of b. In the case of sine and cosine functions, the length of the fundamental cycle or the period becomes $2\pi / |b|$.

(c) The basic graph is translated so $(0,0) \to (h,k)$, the whole graph moves accordingly.

Example 1

Sketch the graph of $y = 2\sin x$ on the interval $[-\pi, 3\pi]$

<u>Solution</u> The basic graph $y = \sin x$ is vertically stretched by a factor of 2, the amplitude becomes 2. The period remains 2π.

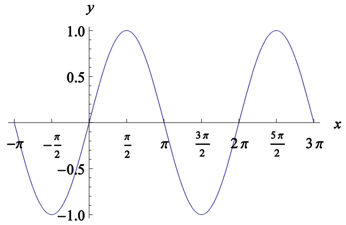


Figure 6.4a $y = \sin x$

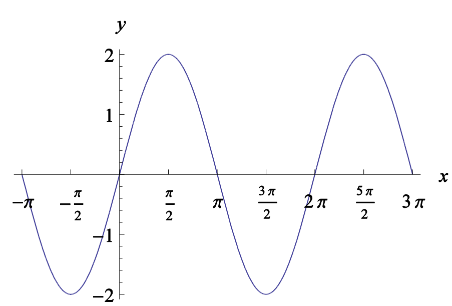


Figure 6.4b $y = 2\sin x$

As noted in Chapter 3, we can avoid what may be complicated applications of the stretching and shrinking rules for transformations by plotting the intercepts with the axes, in particular the intercepts of the fundamental cycle with the $x-axis$. We can also include, for sine and cosine, the maximum and minimum points of the fundamental cycle as shown in Method 2 of Example 2.

Example 2

Sketch the graph of $y = \sin 2x,\ x \in [-\pi, 2\pi]$

Solution

Method 1: Using the shrinking and stretching rules, the basic graph of $y = \sin x$ is shrunk horizontally so the period becomes $\frac{2\pi}{2} = \pi$. Shrink the fundamental cycle to period π (Figure 6.5a) and then extend the graph in either direction (Figure 6.5b).

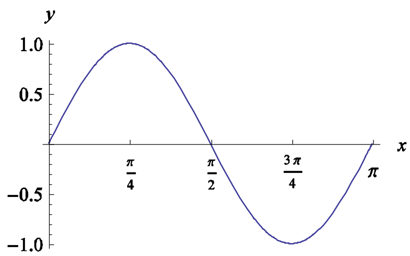


Figure 6.5a $y = \sin 2x,\ [0, \pi]$

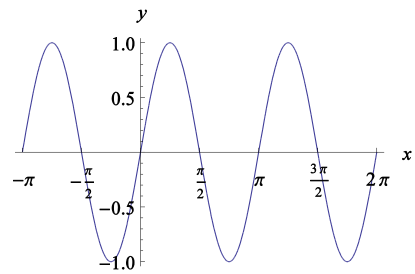


Figure 6.5b $y = \sin 2x,\ [-\pi, 2\pi]$

Method 2: Using the intercepts and turning points for the fundamental cycle, we argue as follows. The values for the fundamental cycle of $\sin\theta$, namely,

θ	0	$\pi/2$	π	$3\pi/2$	2π
$\sin\theta$	0	1	0	-1	0

become these values for the fundamental cycle of $\sin 2x$ by putting $\theta = 2x \Rightarrow x = \frac{\theta}{2}$.

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\sin 2x$	0	1	0	-1	0

Simply draw the fundamental cycle of $y = \sin 2x$ using these points, as in Figure 6.5a, and then extend it as in Figure 6.5b.

Example 3

Sketch the graphs of $y = \frac{1}{2}\cos x$, $y = 3\cos x,\ x \in [-\pi, 3\pi]$.

Solution The basic graph is that of $y = \cos x$. We need to vertically shrink it for $y = \frac{1}{2}\cos x$ (amplitude $\frac{1}{2}$) and vertically stretch it for $y = 3\cos x$ (amplitude 3).

The intercepts do not change.

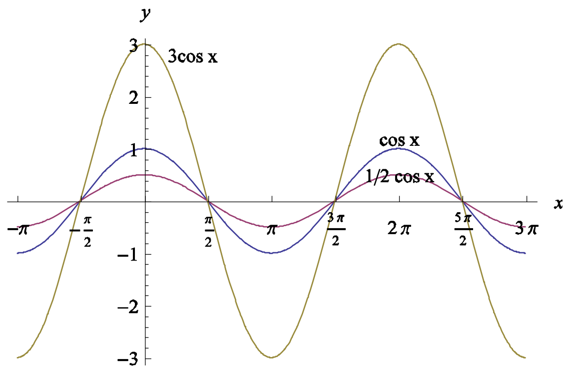


Figure 6.6 $y = \cos x,\ y = \frac{1}{2}\cos x,\ y = 3\cos x$

Example 4

Sketch $y = 3\sin\frac{1}{2}x,\ \ x \in [-2\pi, 4\pi]$

Solution

Method 1: The basic graph is $y = \sin x$. The new graph $y = 3\sin\frac{1}{2}x$ is stretched vertically to have amplitude 3 and horizontally to have period $\frac{2\pi}{1/2} = 4\pi$.

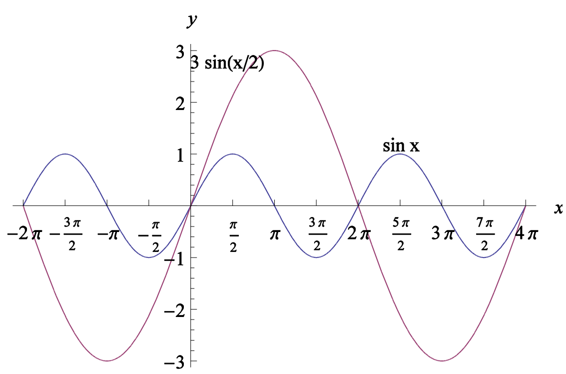


Figure 6.7 $y = \sin x,\ y = 3\sin\frac{1}{2}x$

Method 2: The values for the fundamental cycle of $3\sin\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
3 $\sin\theta$	0	3	0	-3	0

become these values for the fundamental cycle of $3\sin\frac{x}{2}$ by putting $\theta = x/2 \Rightarrow x = 2\theta$.

x	0	π	2π	3π	4π
3 $\sin 1/2x$	0	3	0	-3	0

So we simply draw the new fundamental cycle and then extend.

Example 5 (Reflections)

Sketch $y=-2\sin x,\ x\in[-\pi,3\pi]$

<u>Solution</u> The basic graph is $y=\sin x$. The new graph $y=-2\sin x$ has amplitude 2 and period 2π and is reflected in the $x-axis$.

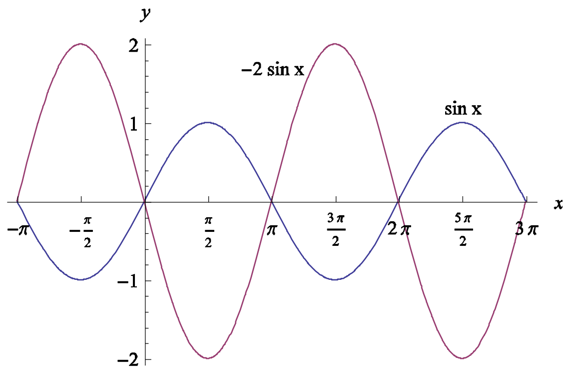


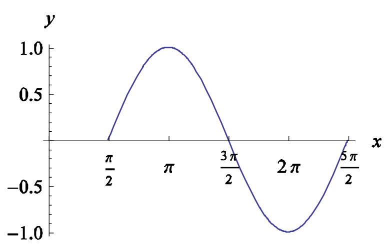
Figure 6.8 $y=\sin x,\ y=-2\sin x$

Example 6 (Translations)

Sketch $y=\sin(x-\dfrac{\pi}{2}),\ x\in[-\pi/2,3\pi]$

<u>Solution</u>

Method 1: Comparing with $y-k=af(b(x-h))$, the required graph is a shift of $y=\sin x$ so that $(0,0)\to(\pi/2,0)$ and the rest of the fundamental cycle shifts accordingly. Draw the repositioned fundamental cycle (Fig. 6.9), and then fill in the rest (Fig. 6.10).



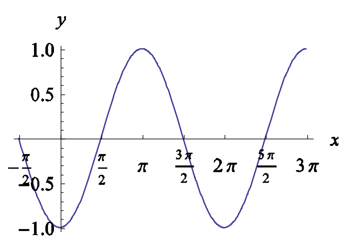


Figure 6.9 $y=\sin(x-\pi/2)$ Figure 6.10 $y=\sin(x-\dfrac{\pi}{2}),\ [-\pi/2,3\pi]$

Method 2:

The values for the fundamental cycle of $\sin\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
$\sin\theta$	0	1	0	-1	0

become these values for the fundamental cycle of $\sin(x-\pi/2)$ by putting $\theta=x-\pi/2$.

x	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$
$\sin(x-\pi/2)$	0	1	0	-1	0

Use this data to draw the fundamental cycle (Figure 6.9) and then extend to Figure 6.10.

Example 7

Sketch $y = 2\cos(3x + \pi), \quad x \in [-\pi, \pi]$

<u>Solution</u>

Method 1: The basic graph is $y = \cos x$. Rewrite as $y = 2\cos[3(x + \frac{\pi}{3})]$. This graph has amplitude 2, period $\frac{2\pi}{3}$ and is translated so that $(0,0) \to (-\frac{\pi}{3}, 0)$ or the whole graph is translated $\frac{\pi}{3}$ to the left. It is then stretched by a factor of 3. The five graphs show the steps to be taken, beginning with $\cos x$.

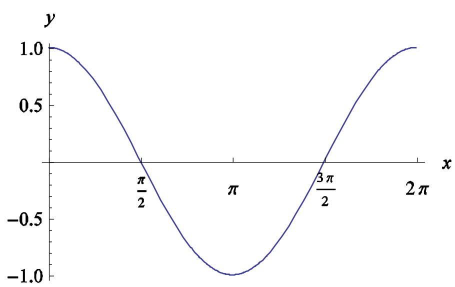


Figure 6.11 $y = \cos x$

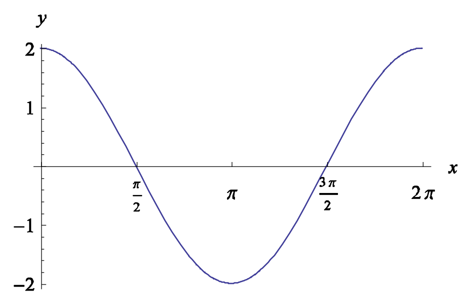


Figure 6.12 $y = 2\cos x$

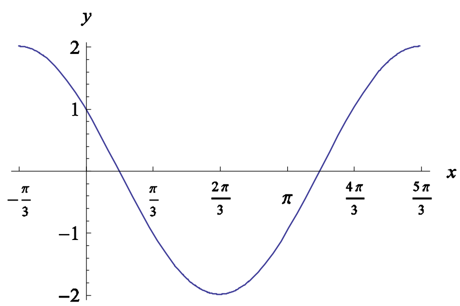


Figure 6.13 $y = 2\cos(x + \frac{\pi}{3})$

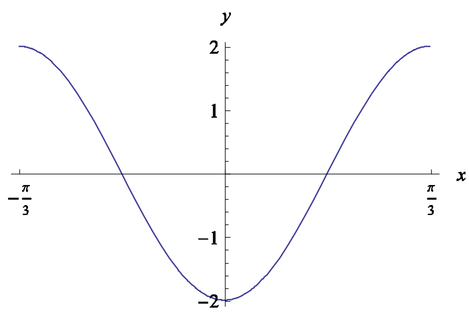


Figure 6.14 $y = 2\cos 3(x + \pi/3), \quad [-\pi/3, \pi/3]$

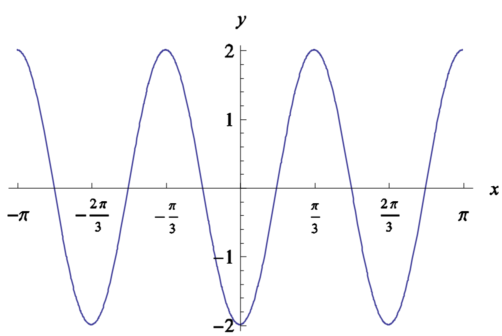


Figure 6.15 $y = 2\cos(3x + \pi), \quad [-\pi, \pi]$

Method 2: (is much simpler!)
The values for the fundamental cycle of $\cos\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
$\cos\theta$	1	0	-1	0	1

become these values for the fundamental cycle of $2\cos(3x+\pi)$ by putting $\theta = 3x+\pi$.

x	$-\pi/3$	$-\pi/6$	0	$\pi/6$	$\pi/3$
$2\cos(3x+\pi)$	2	0	-2	0	2

Draw the fundamental cycle of $y = 2\cos(3x+\pi)$ as shown in Figure 6.14 and then extend it to get Figure 6.15.

Example 8
Sketch $y = \sin x + 2,\ x \in [-2\pi, 2\pi]$

Solution Write as $y - 2 = \sin x$. The basic graph of $y = \sin x$ which oscillates around the line $y = 0$ (the $x - axis$) is translated so $(0,0) \to (0,2)$.
Accordingly, the new graph oscillates around the line $y = 2$, but otherwise has the same shape (period and amplitude) as $y = \sin x$.

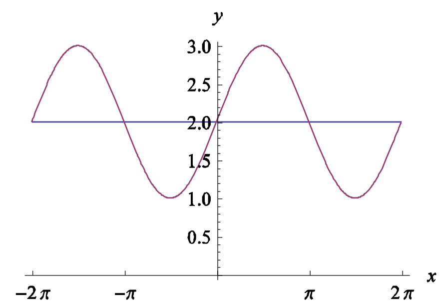
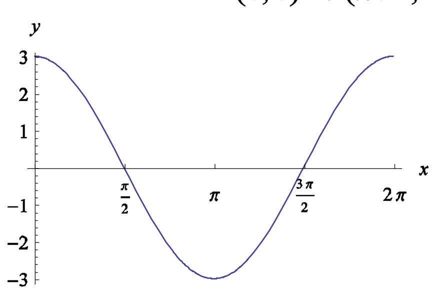


Figure 6.16 $y = \sin x + 2$

Example 9
Sketch $y = 1 + 3\cos(2x - \pi),\quad x \in [0, 2\pi]$
Solution

Method 1: Rewrite as $y - 1 = 3\cos[2(x - \dfrac{\pi}{2})]$. The basic graph is $y = \cos x$ with amplitude 1 and period 2π. First, the new graph is stretched vertically to amplitude 3 and translated so that $(0,0) \to (\pi/2, 1)$ to become $y = 1 + 3\cos(x - \pi/2))$.



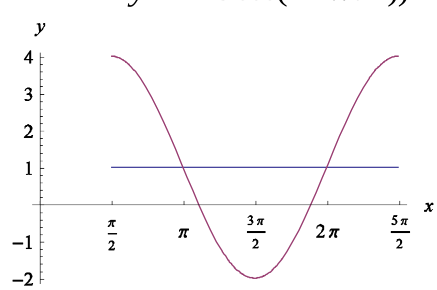
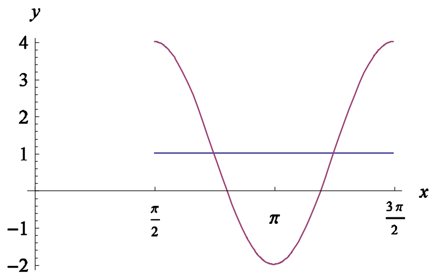


Figure 6.17 $y = 3\cos x$ Figure 6.18 $y = 1 + 3\cos(x - \pi/2))$

Then it is shrunk horizontally to period $\dfrac{2\pi}{2} = \pi$. Finally we fill in the required remainder of the domain $[0, 2\pi]$.



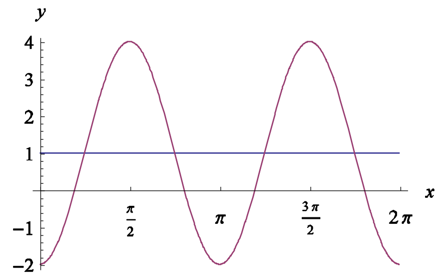


Figure 6.19 $y = 1 + 3\cos\left[2(x - \pi/2)\right]$ Figure 6.20 $y = 1 + 3\cos\left[2(x - \pi/2)\right], [0, 2\pi]$.

Method 2:
The values for the fundamental cycle of $3\cos\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
3 $\cos\theta$	3	0	-3	0	3

become these values for the fundamental cycle of $3\cos(2x - \pi)$ by putting $\theta = 2x - \pi$.

x	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
$3\cos(2x - \pi)$	3	0	-3	0	3

Draw the fundamental cycle of $y = 3\cos(2x - \pi)$ and then elevate it by 1 unit (shift the origin to $(0,1)$) to give Figure 6.19 and extend it to get Figure 6.20.

6.3 Graph of $\tan\theta$

Let us consider $\tan\theta = \dfrac{y}{x}$ for $0 \le \theta \le \dfrac{\pi}{2}$.

At $\theta = 0,\ y = 0,\ x = 1$, so $\tan\theta = 0$

At $\theta = \dfrac{\pi}{2},\ y = 1,\ x = 0$, so $\tan\theta$ is undefined.

As θ increases from 0 *to* $\dfrac{\pi}{2}$, y increases towards 1 and x decreases towards 0.

Let us consider various values of the points (x, y) on the unit circle $x^2 + y^2 = 1$ as $x \to 0,\ y \to 1$ and the consequential values of $\tan\theta$.

y	0	0.5	0.9	0.99	0.999	0.9999	0.99999
x	1	0.86603	0.43589	0.14107	0.04471	0.01414	0.00447
$\tan\theta$	0	0.57735	2.06472	7.01792	22.3439	70.7054	223.605

Graphically we would get the following by plotting θ and $\tan\theta$.

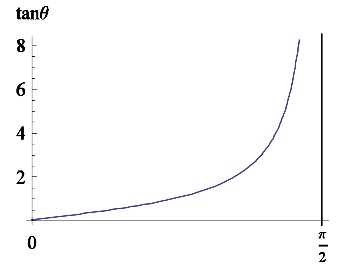


Figure 6.21 $y = \tan\theta,\ 0 \le \theta \le \pi/2$

We say $\tan\theta$ approaches the line $\theta = \dfrac{\pi}{2}$ and that the line $\theta = \dfrac{\pi}{2}$ is a vertical asymptote. The same happens in the intervals $(\pi/2, \pi), (\pi, 3\pi/2)$ and so on, except $\tan\theta$ alternates between being positive and negative (ASTC). Let's switch the independent variable to x for the function $y = \tan x$. The graph is shown in Figure 6.22.

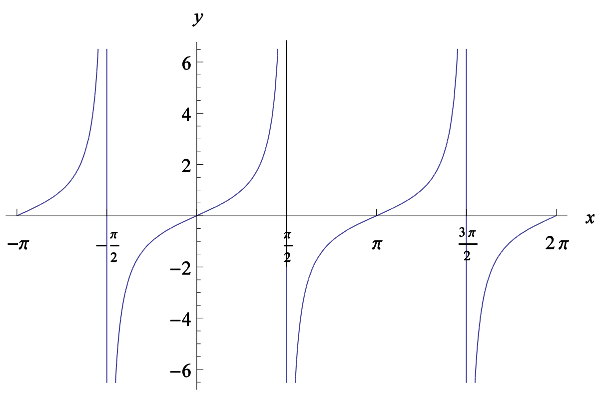


Figure 6.22 $y = \tan x,\ -\pi/2 \le x \le 2\pi$

Summary

The domain of $y = \tan x$ is $\left\{ x : x \ne \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \right\}$.

The range of $y = \tan x$ is $(-\infty, \infty)$.

The period of $y = \tan x$ is π.

Example 10

Sketch $y = 5\tan 2x,\ \ x \in [-\pi, \pi]$

Solution

Method 1: The basic graph is that of $y = \tan x$, with a vertical stretching of 5 and a period of $\frac{\pi}{2}$ due to the shrinking factor 2. The vertical stretching and horizontal shrinking takes $(\frac{\pi}{4},1) \to (\frac{\pi}{8},5)$.

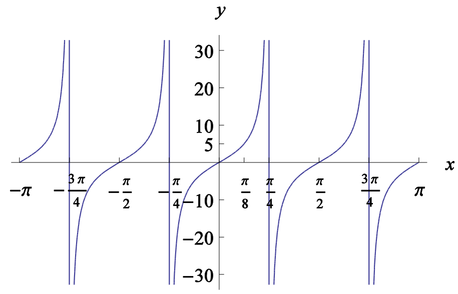


Figure 6.23 $y = 5\tan 2x, \ [-\pi, \pi]$

Method 2:

The fundamental cycle of $y = \tan\theta$ may be summarized as follows.

θ	$-\pi/2^+$	$-\pi/4$	0	$\pi/4$	$\pi/2^-$
$y = \tan\theta$	$-\infty$	-1	0	1	∞

Putting $\theta = 2x$ and multiplying by 5, the fundamental cycle of $y = 5\tan 2x$ becomes,

x	$-\pi/4^+$	$-\pi/8$	0	$\pi/8$	$\pi/4^-$
$y = 5\tan 2x$	$-\infty$	-5	0	5	∞

Draw the fundamental cycle as shown in Figure 6.23 and then extend it.

Exercises 6A

Draw the graphs of the following functions. State the amplitude, period and phase shift (if any). Show two full periods.

1. $f(x) = \sin 2x$
2. $y = \cos\frac{x}{2}$
3. $y = \tan 3x$
4. $y = 2\cos 2x$
5. $g(x) = \sin\frac{x}{3}$
6. $g(x) = \cos(-4x)$
7. $y = \frac{1}{4}\sin x$
8. $y = 4\cos x$
9. $y = \sin 4x$
10. $y = \sin(x - 2\pi)$
11. $y = 4\cos(x + \frac{\pi}{4})$
12. $y = 2\cos x - 3$
13. $f(x) = 1 + \cos x$

14. $f(x) = -\sin x$
15. $y = -3\sin 3x$
16. $y = -\frac{1}{3}\cos\frac{1}{3}x$
17. $y = 3\cos(x + \frac{\pi}{4})$
18. $y = -4\sin 2(x + \frac{\pi}{2})$
19. $y = 1 + \cos(3x + \frac{\pi}{2})$
20. $y = \tan(x - \frac{\pi}{4})$
21. $y = \tan 4x$
22. $y = \sec 2x$

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6.5 The Reciprocal Trigonometric Functions and Their Graphs

6.5.1 Definitions

The cosecant, secant and cotangent trigonometric functions are defined as the reciprocals of the sine, cosine and tangent functions:

$$\csc x = \frac{1}{\sin x},\ \sec x = \frac{1}{\cos x},\ \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

To evaluate these functions at particular values of x, first evaluate sine, cosine or tangent of x and invert the result.

Example 11

Evaluate $\csc\frac{\pi}{2}$, $\sec\frac{\pi}{2}$, $\cot\frac{\pi}{6}$, $\csc\frac{11\pi}{3}$, $\sec 5\pi$, $\cot 2\pi$.

Solutions

$$\csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$$

$$\sec\frac{\pi}{2} = \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0} \quad \textit{undefined}$$

$$\cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

$$\csc\frac{11\pi}{3} = \frac{1}{\sin(4\pi - \frac{\pi}{3})} = \frac{1}{\sin(-\frac{\pi}{3})} = -\frac{1}{\sqrt{3}/2} = -\frac{2\sqrt{3}}{3}$$

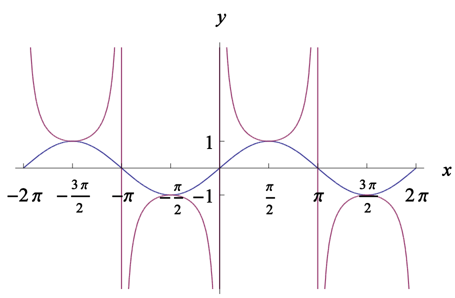
$$\sec 5\pi = \frac{1}{\cos 5\pi} = \frac{1}{\cos(\pi + 4\pi)} = \frac{1}{\cos\pi} = \frac{1}{-1} = -1$$

$$\cot 2\pi = \frac{1}{\tan 2\pi} = \frac{1}{0} \quad \textit{undefined}$$

6.5.2 Graphs of $\csc x, \sec x$

As reciprocals of sine and cosine, the cosecant and secant functions have period 2π, and as the reciprocal of the tangent function, the cotangent function has period π. Since $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$, both have the same undefined points so both have the same vertical asymptotes at $x = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$. Since $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, both have the same vertical asymptotes where $\sin x = 0$, namely $x = n\pi,\ n \in \mathbb{Z}$.
To draw the graphs of $y = \csc x,\ y = \sec x$, first draw the graphs of $y = \sin x,\ y = \cos x$ respectively, and then the asymptotes at $x = n\pi,\ n \in \mathbb{Z}$ and $x = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$ respectively.

When we plot points, the sine and cosecant curves meet wherever $\sin x = 1$ since then $\csc x = 1$ and the cosine and secant curves meet wherever $\cos x = 1$ since then $\sec x = 1$ but otherwise since $-1 < \sin x < 1$, $\csc x > 1$ *or* < -1, and since $-1 < \cos x < 1$, $\sec x > 1$ *or* < -1. So we get the graphs shown in Figures 6.24 and 6.25.



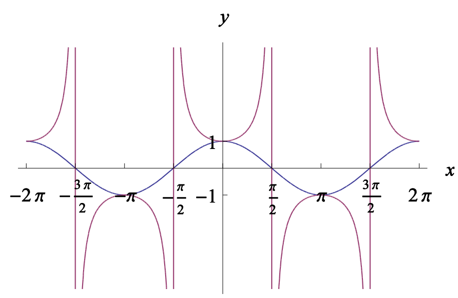


Figure 6.24 $y = \csc x$ "*above*" $y = \sin x$ Figure 6.25 $y = \sec x$ "*above*" $y = \cos x$

Example 12
Sketch $y = 3\sec(x - \pi),\ [-2\pi, 2\pi]$
<u>Solution</u>
Method 1: The basic graph $y = \sec x$ has maxima and minima at $y = \pm 1$ and period 2π. The graph of $y = 3\sec(x - \pi)$ is stretched so the maxima and minima are at $y = \pm 3$ and the period is still 2π. It is then translated so $(0,0) \to (\pi, 0)$, that is the fundamental cycle shifts π to the right. See Figure 6.26.

Method 2: The fundamental cycle of $y = \sec\theta$ may be summarized as follows.

θ	$-\pi/2^+$	0	$\pi/2^-$	$\pi/2^+$	π	$3\pi/2^-$
$y = \sec\theta$	∞	1	∞	$-\infty$	-1	$-\infty$

Putting $x = \pi + \theta$, the fundamental cycle of $y = 3\sec(x - \pi)$ becomes,

x	$\pi/2^+$	π	$3\pi/2^-$	$3\pi/2^+$	2π	$5\pi/2^-$
$y = \sec(x - \pi)$	∞	1	∞	$-\infty$	-1	$-\infty$

Draw the fundamental cycle and then extend it as in Figure 6.26.

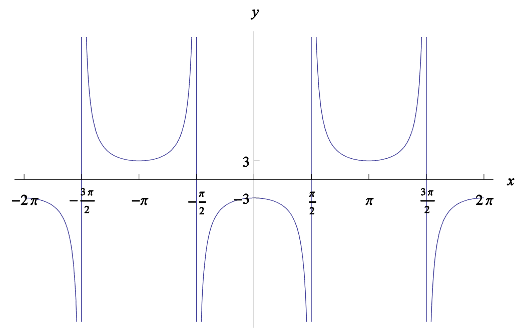
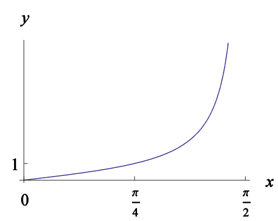


Figure 6.26 $y = 3\sec(x - \pi),\ [-2\pi, 2\pi]$

6.5.3 Graph of $\cot x$

Since $\cot x = \dfrac{1}{\tan x}$ it follows that if $\tan x \to 0$ then $\cot x \to \infty$ and if $\tan x \to \infty$ then $\cot x \to 0$. The vertical asymptotes are at the values of x for which $\tan x = 0$, namely $x = n\pi$. For $[0, \dfrac{\pi}{2}]$ we have the two graphs,



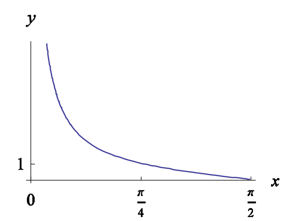


Figure 6.27 $y = \tan x,\ 0 \le x < \pi/2$

Figure 6.28 $y = \cot x,\ 0 < x \le \pi/2$

The complete graph of $y = \cot x,\ [-2\pi, 2\pi]$ is shown in Figure 6.29.

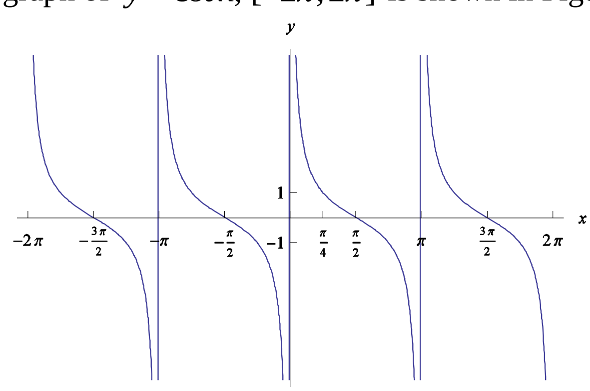


Figure 6.29 $y = \cot x,\ [-2\pi, 2\pi]$

6.6 Inverse Trigonometric Functions and their Graphs

6.6.1 The Problem

Recall that a function f has an inverse f^{-1} if and only if it is one-to-one, that is each value of the dependent variable (often y) corresponds to exactly one value of the independent variable (often x).

Graphically, the inverse function f^{-1} exists if and only if f satisfies the horizontal line test, namely, any line parallel to the $x-axis$ intersects the graph of f at most once.

In Section 2.7.6 we discussed functions that fail the horizontal line test for an unrestricted domain. If we restricted the domain, however, we could find an inverse function. Clearly, the graphs of all trigonometric functions fail the horizontal line test big time due to their periodicity. Any horizontal line in their ranges will intersect them an infinite number of times.

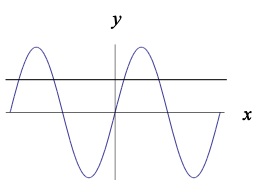


Figure 6.30 $y = \sin x$

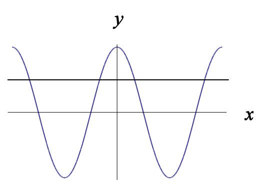


Figure 6.31 $y = \cos x$

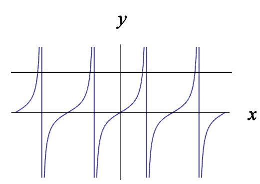


Figure 6.32 $y = \tan x$

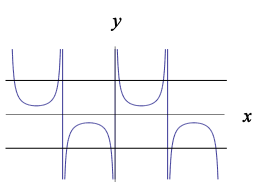


Figure 6.33 $y = \csc x$

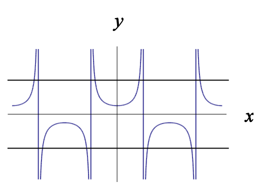


Figure 6.34 $y = \sec x$

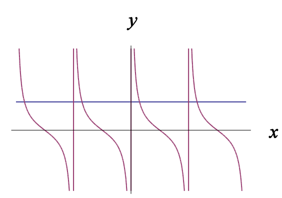


Figure 6.35 $y = \cot x$

What we need to do in each case is to select a part of the graph that includes the origin and that satisfies the horizontal line test. That part will then have an inverse whose graph is obtained by reflection in the line $y = x$.

6.6.2 The Inverse Sine Function

The sine graph obeys the horizontal line test on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We write the inverse sine function as $y = \sin^{-1} x$ or as $y = \arcsin x$.

Since it is the inverse of $y = \sin x$, we can also write it as $x = \sin y$. That is,

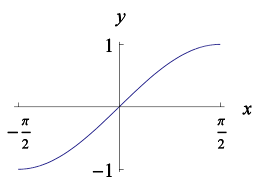
$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

Then, restricting the domain of $y = \sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have,

Definition

The inverse of $y = \sin x$, with domain $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $y \in [-1, +1]$ is called $y = \sin^{-1} x$ or $y = \arcsin x$ with domain $x \in [-1, +1]$ and range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The graphs are as shown in Figures 6.36 and 6.37.



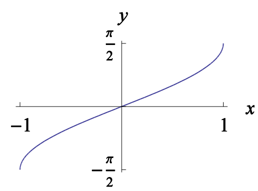


Figure 6.36 $y = \sin x, [-\pi/2, \pi/2]$ Figure 6.37 $y = \sin^{-1} x, [-1,1]$

6.6.3 The Inverse Cosine Function

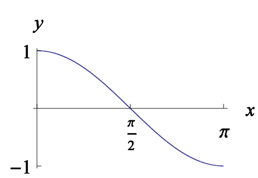
The cosine graph obeys the horizontal line test on the domain $[0, \pi]$. We write the inverse cosine function as $y = \cos^{-1} x$ or $y = \arccos x$. Since it is the inverse of $y = \cos x$, we can also write it as $x = \cos y$. That is,

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

Then, restricting the domain of $y = \cos x$ to $[0, \pi]$, we have,

Definition

The inverse of $y = \cos x$, with domain $x \in [0, \pi]$ and range $y \in [-1, +1]$ is called $y = \cos^{-1} x$ or $y = \arccos x$ with domain $x \in [-1, +1]$ and range $y \in [0, \pi]$.
The graphs are as shown in the figures.



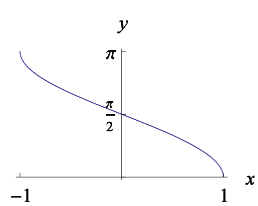


Figure 6.38 $y = \cos x, \ [0, \pi]$ Figure 6.39 $y = \cos^{-1} x, \ [-1,1]$

6.6.4 The Inverse Tangent Function

The tangent graph obeys the horizontal line test on the domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We write the inverse tangent function as $y=\tan^{-1}x$ or as $y=\arctan$. Since it is the inverse of $y=\tan x$, we can also write it as $x=\tan y$. That is,

$$y=\tan^{-1}x \Leftrightarrow x=\tan y$$

Then, restricting the domain of $y=\tan x$ to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we have,

Definition

The inverse of $y=\tan x$, with domain $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and range $y\in(-\infty,+\infty)$ is called $y=\tan^{-1}x$ or $y=\arctan x$ with domain $x\in[-\infty,+\infty]$ and range $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

The graphs are as shown in the figures.

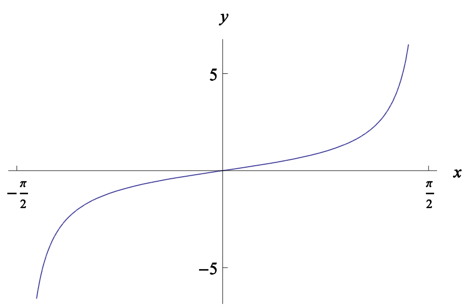


Figure 6.40 $y=\tan x,[-\pi/2,\pi/2]$

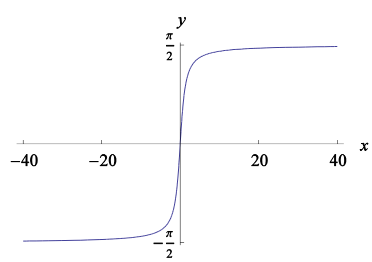


Figure 6.41 $y=\tan^{-1}x,\ (-\infty,\infty)$

6.6.5 Evaluating Inverse Trigonometric Functions

In words, the solution x to $y=\sin^{-1}x \Leftrightarrow x=\sin y$ is " x is the angle whose sine is y ".

Note and note again, that x is an angle. In the case of $\sin^{-1}$ and $\tan^{-1}$, $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, in the case of $\cos^{-1}$ we must have $x\in[0,\pi]$.

Let us stress again, your answer must be in the allowable domain of the particular inverse function you are dealing with. Consider the following examples.

Example 13

Evaluate $(a)\ y=\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ $(b)\ y=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $(c)\ y=\tan^{-1}1$

<u>Solution</u>

(a) The angle whose sine is $\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Then $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$ since this value lies in the allowable domain $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(b) $\frac{\pi}{6}\in[0,\pi]$

(c) $\frac{\pi}{4}\in[-\frac{\pi}{2},\frac{\pi}{2}]$

Example 14

Evaluate $(a)\ y=\sin^{-1}\left(-\frac{1}{2}\right)$ $(b)\ y=\cos^{-1}(-1)$, $(c)\ y=\tan^{-1}(-1)$

<u>Solution</u>

We need to be careful, inverse functions have a restricted domain.

(a) The angle whose sine is $\frac{1}{2}$ is $\frac{\pi}{6}$. The angle whose sine is $-\frac{1}{2}$ is in the third or fourth quadrants, given by $\sin\left(\pi+\frac{\pi}{6}\right)$, $\sin\left(2\pi-\frac{\pi}{6}\right)$ *or* $\sin\left(-\frac{\pi}{6}\right)$. We choose the answer $-\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

(b) The angle whose cosine is -1 is π *or* $-\pi$. We choose $\pi\in[0,\pi]$.

(c) The angle whose tangent is 1 is $\frac{\pi}{4}$. The angle whose tangent is -1 is in the second or fourth quadrants, given by $\tan\left(\pi+\frac{\pi}{4}\right)$, $\tan\left(2\pi-\frac{\pi}{4}\right)$ *or* $\tan\left(-\frac{\pi}{4}\right)$. We choose the answer $-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Example 15

Evaluate $(a)\sin(\sin^{-1}(-\sqrt{3}/2)$ $(b)\sin^{-1}(\sin(-\pi/6)$ $(c)\ \sin^{-1}(\sin(5\pi/6)$

<u>Solution</u>

$(a)\ \sin^{-1}(-\sqrt{3}/2)=-\pi/3\in[-\pi/2,\pi/2]$

$\sin(-\pi/3)=-\sqrt{3}/2$

$(b)\ \sin(-\pi/6)=-1/2$

$\sin^{-1}(-1/2)=-\pi/6$

$(c)\ \sin(5\pi/6)=\sin(\pi-\pi/6)=\sin(\pi/6)=1/2$

$\sin^{-1}(1/2)=\pi/6\in[-\pi/2,\pi/2]$

Example 16

Evaluate $(a)\ \cos^{-1}(\tan(5\pi/4))$ $(b)\ \sin^{-1}(\cos(3\pi/4))$ $(c)\ \sin(\tan^{-1}(-1))$

Solution

$(a)\ \tan(5\pi/4)=\tan(\pi+\pi/4)=\tan(\pi/4)=1$

$\cos^{-1}(1)=0\in[0,\pi]$

$(b)\ \cos(3\pi/4)=\cos(\pi-\pi/4)=-\cos(\pi/4)=-\sqrt{2}/2$

$\sin^{-1}(-\sqrt{2}/2)=-\pi/4\in[-\pi/2,\pi/2]$

$(c)\ \tan^{-1}(-1)=-\pi/4\in[-\pi/2,\pi/2]$

$\sin(-\pi/4)=-\sin(\pi/4)=-\sqrt{2}/2$

Example 17

Evaluate $(a)\ \tan(\sin^{-1}(4/5)$ $(b)\ \cos(\sin^{-1}(2/3))$ $(c)\ \sin(\cos^{-1}(-1/3))$

Solution

(a) We draw a right triangle for the angle θ whose sine is $4/5$ and calculate the third side – see figure 6.42. Then $\tan\theta=4/3$

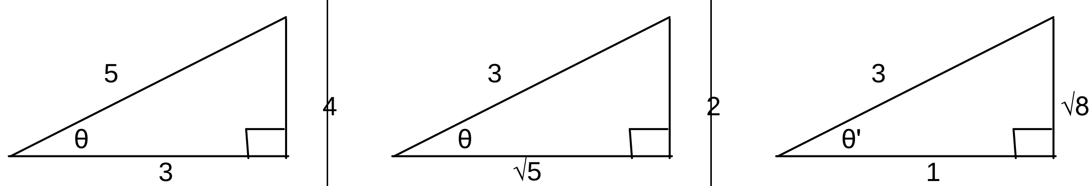


Figure 6.42 Figure 6.43 Figure 6.44

(b) See Figure 6.43. Then $\cos\theta=\sqrt{5}/3$

(c) See Figure 6.44. In this case we use the reference angle θ' in the right triangle and argue as follows. If $\theta=\cos^{-1}(-1/3)$ and $\theta\in[0,\pi]$ then θ must be in the second quadrant where sine is positive. Then $\sin\theta=\sin\theta'=\sqrt{8}/3=2\sqrt{2}/3$

6.6.6 Graphs of inverse cosecant, secant and cotangent functions

The remaining inverse trigonometric functions have the following graphs.

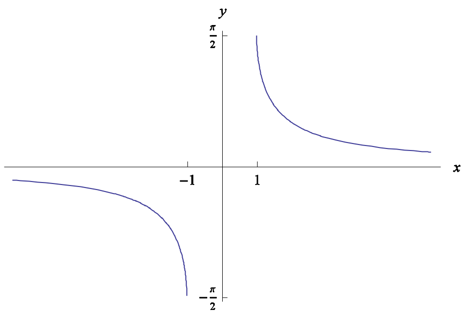


Figure 6.45 $y = \csc^{-1} x$

Domain = $\{x \mid x \le -1 \text{ or } x \ge 1\}$

Range = $\left\{ y \mid -\frac{\pi}{2} \le y < 0 \text{ or } 0 < y \le \frac{\pi}{2} \right\}$

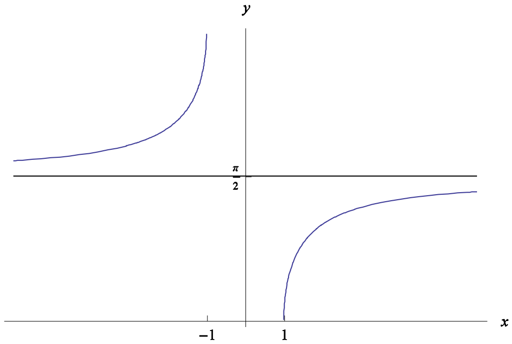


Figure 6.46 $y = \sec^{-1} x$

Domain = $\{x \mid x \le -1 \text{ or } x \ge 1\}$

Range = $\left\{ y \mid 0 \le y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y \le \pi \right\}$

As an exercise, use the reflection principle to sketch the graph of $y = \cot^{-1}(x)$ and state its domain and range.

Exercises 6B

1. Graph $y = \frac{1}{3}\csc x$. Show 2 full periods.
2. Graph $y = 2\sec 2x$. Show 2 full periods.
3. Graph $y = 2\cot(x + \frac{\pi}{2})$. Show 2 full periods.

Find the exact values of:

4. $\sin^{-1}\frac{1}{2}$
5. $\sin^{-1}\frac{\sqrt{2}}{2}$
6. $\sin^{-1}\left(-\frac{1}{2}\right)$
7. $\cos^{-1}\frac{\sqrt{3}}{2}$
8. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
9. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
10. $\cos^{-1}1$
11. $\tan^{-1}\frac{\sqrt{3}}{3}$
12. $\tan^{-1}0$
13. $\tan^{-1}(-\sqrt{3})$
14. $\sin^{-1}(\sin\frac{\pi}{3})$
15. $\sin^{-1}(\sin\frac{5\pi}{6})$
16. $\tan(\tan^{-1}125)$
17. $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$
18. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$
19. $\sin^{-1}(\sin\pi)$

Use a sketch to the find the exact values of:

20. $\cos\left(\sin^{-1}\frac{4}{5}\right)$
21. $\tan\left(\cos^{-1}\frac{5}{13}\right)$
22. $\tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

23. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

24. $\sec\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$

25. $\tan\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$

26. $\csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$

Chapter 7

Right-Angle Trigonometry

7.1 Description

A major application of trigonometry is in the solution of right-angle triangles. This means, given some of the sides and angles, find the others. We will measure angles in degrees only. If we call an angle A, we mean $A = A^0$. We can label any right-angle triangle as shown in Figure 7.1.

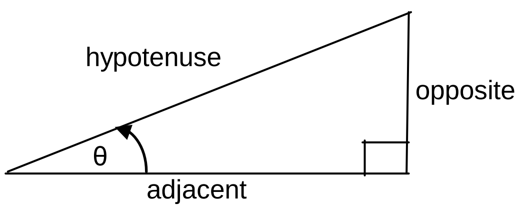


Figure 7.1

We can then superimpose this triangle on the unit circle as shown in Figure 7.2.

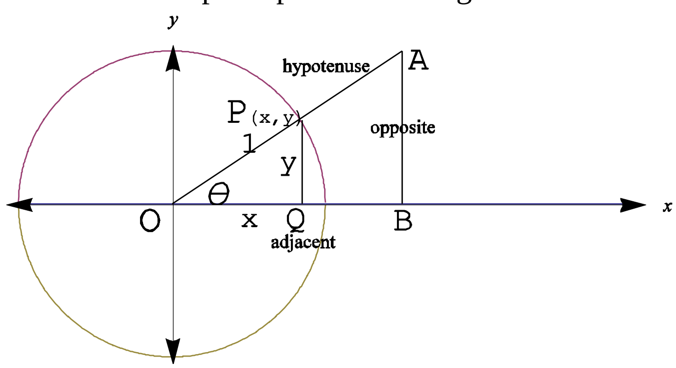


Figure 7.2

The triangles OPQ and OAB are similar triangles. They have the same angles $\theta, 90^0 - \theta, 90^0$ and therefore their corresponding sides are in the same ratio. Then,

$$\frac{y}{1} = \frac{opposite\ AB}{hypotenuse\ OA} \Rightarrow \sin\theta = \frac{opposite}{hypotenuse},$$

$$\frac{x}{1} = \frac{adjacent\ OB}{hypotenuse\ OA} \Rightarrow \cos\theta = \frac{adjacent}{hypotenuse}, -$$

$$\frac{y}{x} = \frac{opposite\ AB}{adjacent\ OB} \Rightarrow \tan\theta = \frac{opposite}{adjacent}.$$

Example 1

Find the six trigonometric values of the angle θ in the triangle

Solution By the Pythagorean Theorem,

the hypotenuse $= \sqrt{3^2+4^2} = 5$. We have

$adj = 3, opp = 4, hypot = 5$.

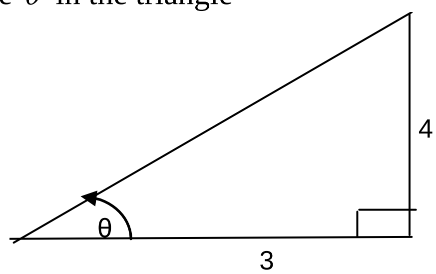


Figure 7.3

The six values are,

$\sin\theta = 4/5, \cos\theta = 3/5, \tan\theta = 4/3, \csc\theta = 5/4, \sec\theta = 5/3, \cot\theta = 3/4$

7.2 Using a Calculator

You can use the [sin], [cos], [tan] and $\boxed{\frac{1}{x}}$ keys on a calculator to find the trigonometric values of any angle.

Example 2

Find $\sin 398^0$

Solution

Type 398, press [sin] key. Answer 0.615661475….

Example 3

Find $\sec 5^0 40'12''$

Solution $\sec 5^0 40'12'' = \sec\left(5^0 + \frac{40}{60}^0 + \frac{12}{3600}^0\right) = \sec 5.67^0$

Type 5.67, press [cos] key, press $\boxed{\frac{1}{x}}$ key. Answer 1.0049

7.3 Solving a right triangle

By convention, we mostly label triangles with vertices A, B C, opposite sides a,b,c and angles A, B, C.

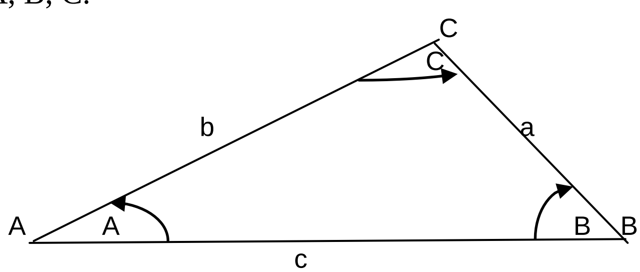


Figure 7.4

Example 4
Solve the triangle,

<u>Solution</u>

$$\sin 40^0 = \frac{a}{5} \Rightarrow a = 5\sin 40^0 = 3.2$$

$$\cos 40^0 = \frac{b}{5} \Rightarrow b = 5\cos 40^0 = 3.8$$

$$B = 90^0 - 40^0 = 50^0$$

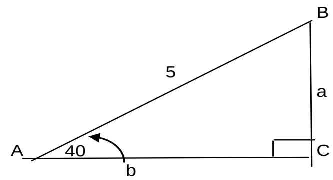


Figure 7.5

Example 5
Find the length of the ramp AB

<u>Solution</u>

$$\sin 18.4^0 = \frac{4'}{c} \Rightarrow c = \frac{4'}{\sin 18.4^0} = 12.7'$$

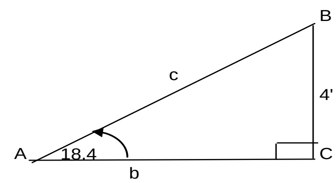


Figure 7.6

Example 6
A surveyor takes the measurements shown in order to calculate the width of a lake at its widest points A and C. Find the width AC.

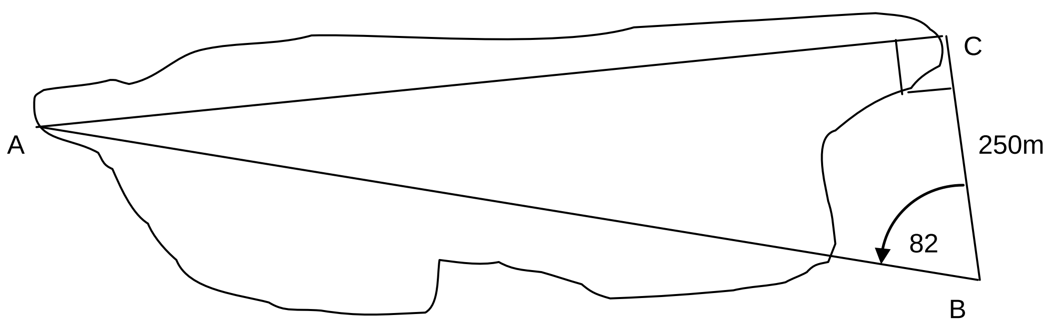


Figure 7.7

<u>Solution</u> $\tan 82^0 = \frac{AC}{250} \Rightarrow AC = 250\tan 82^0 = 1779m$

7.3.1 Definitions of angles of elevation and depression

*The **angle of elevation θ** is the angle through which an observer must elevate his sight from the horizontal in order to focus on a particular point above him.*

*The **angle of depression φ** is the angle through which an observer must lower her eyes from the horizontal in order to focus on a particular point below her.*

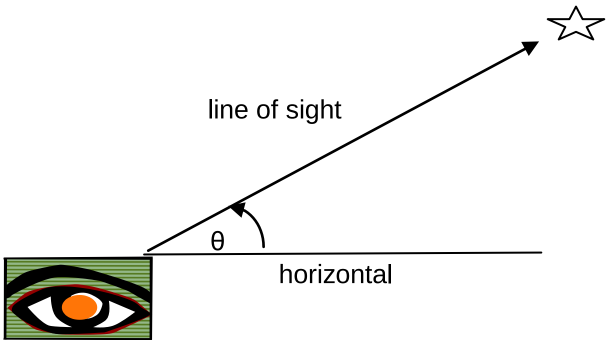


Figure 7.8a Angle of Elevation

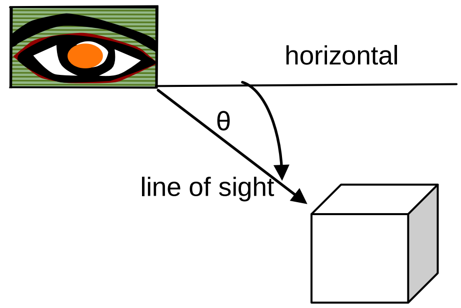


Figure 7.8b Angle of Depression

Example 7
A navy pilot flying at 3000 feet wants to land on an aircraft carrier that she sees at an angle of depression of 15^0 below her. What is the actual distance of the plane from the carrier?
Solution:

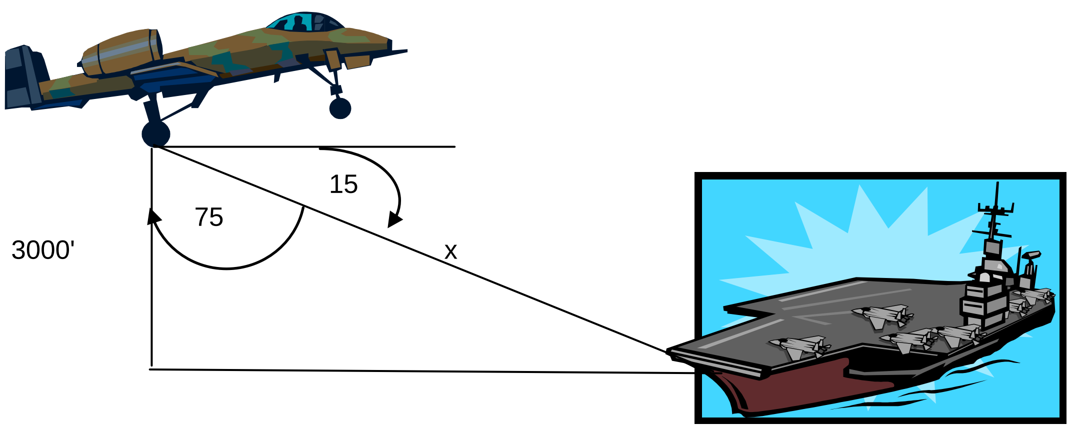


Figure 7.9

$$\cos 75^0 = \frac{3000}{x} \Rightarrow x = \frac{3000}{\cos 75^0} = 11{,}591' = 2.2 \text{ miles.}$$

Example 8
Harry wants to photograph a rocket at its first stage of separation that he knows occurs at 3000 feet. If he is 2 miles away, at what angle should he point his camera?
Solution See Figure 7.10. Note 2 miles = 2x5280 = 10560 feet.

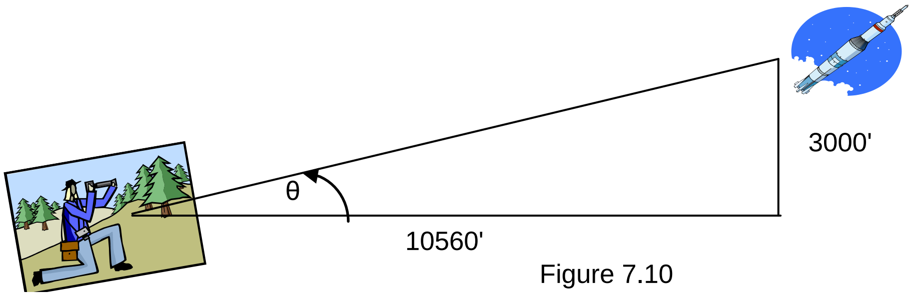


Figure 7.10

$$\tan\theta = \frac{3000}{10560} \Rightarrow \theta = 15.9^0 \text{ (use the } \tan^{-1} \text{ key)}$$

7.4 Area of a triangle

We define area by stating the area of a rectangle is $A = bh$. Clearly the area of the triangle formed by drawing its major diagonal is:

$$A = \frac{1}{2}bh.$$

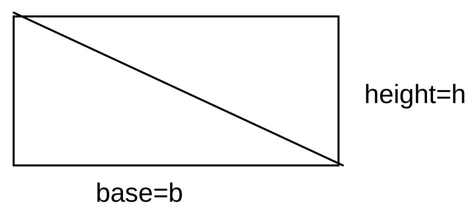


Figure 7.11

Consider the triangle ΔABC below.

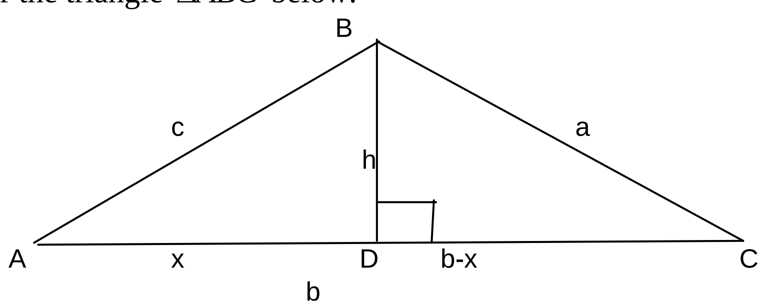


Figure 7.12

The area of the $\Delta BAD = \frac{1}{2}xh$

The area of the $\Delta BCD = \frac{1}{2}(b-x)h$

So the area of the $\Delta ABC = \frac{1}{2}xh + \frac{1}{2}(b-x)h = \frac{1}{2}bh$.

Then, we have a general formula. The area of any triangle is one half of the product of its base by its perpendicular height.

Further, $\sin C = \frac{h}{a} \Rightarrow h = a\sin C$

Hence, the area of the triangle $\Delta ABC = \frac{1}{2}ab\sin C$

We can orient the triangle in three different ways so either A or B or C is the vertex. The resulting formulas are,

Area of the triangle $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

In words, the area of a triangle is one-half the product of the lengths of two sides times the sine of the angle enclosed by them.

Example 9

A circle is inscribed in a hexagon of side 12. Find the area of the region between the hexagon and the circle.

Solution

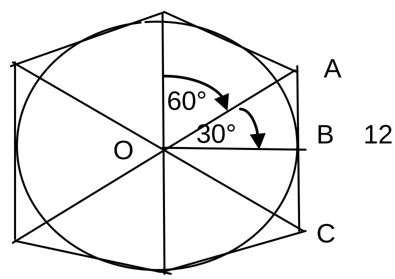


Figure 7.13

Consider one of the six pieces OABC, a sector of the circle inscribed in a triangle. The angles are $\angle AOC = 60^0, \angle AOB = 30^0$. Then,

$\tan 30^0 = \frac{AB}{OB} = \frac{6}{OB} \Rightarrow OB = 10.3923,$

Area $\Delta AOC = \frac{1}{2}bh = \frac{1}{2}\bullet 12 \bullet 10.3923 = 62.3538$

Area of hexagon $= 6 \bullet 62.3538 = 374.1230$

Area of circle $= \pi r^2 = \pi \bullet 10.3923^2 = 339.2917$

Area of region between $= 34.8$

Exercises 7A

In 1 through 5, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean theorem to determine the third side and then find the other 5 trig functions of θ .

1. $\sin\theta = \dfrac{5}{13}$
2. $\cos\theta = \dfrac{5}{6}$
3. $\tan\theta = \dfrac{4}{5}$
4. $\sec\theta = \dfrac{17}{7}$
5. $\csc\theta = 9$
6. Solve the right triangle ABC where $C = 90^0$, $B = 54^0$, $c = 15$
7. At a certain time of day, the angle of elevation of the sun is 40^0. To the nearest foot, find the height of a tree whose shadow is 35 feet long.
8. The Washington Monument is 555 feet high. If you stand a quarter of a mile or 1320 feet from the base of the monument, find the angle of elevation of its top to the nearest degree.
9. A road is inclined at an angle of 5^0. After driving 5000 feet along this road, find the driver's increase in altitude to the nearest foot.
10. You are standing 45 meters from the base of the Empire State building. You estimate the angle of elevation to the top floor is 82^0.
    a. If the total height of the building is another 123 meters above the 86th floor, what is the approximate height of the building.
    b. One of your friends is on the 86th floor. What is the distance between you and your friend?
11. The sun is 25^0 above the horizon. Find the length of the shadow cast by a building that is 100 feet tall.
12. A passenger in an airplane at an altitude of 10 kilometers sees two towns directly east of the plane. The angles of depression of the two towns are 28^0 and 55^0. How far apart are the towns?
13. Find the area of the triangle ABC where $B = 130^0$, $a = 62$, $c = 20$
14. Find the area of the triangle ABC where $A = 5^0 15'$, $b = 4.5$, $c = 22$

==

7.5 Solving any triangle

We can use the law of sines to solve a general triangle (it does not need to be a right triangle) if we are given two angles and any side or two sides and an angle opposite one of them. We can use the law of cosines to solve the other two possibilities for any other general triangle, that is, we are given three sides or two sides and the included angle.

7.5.1 Law of Sines

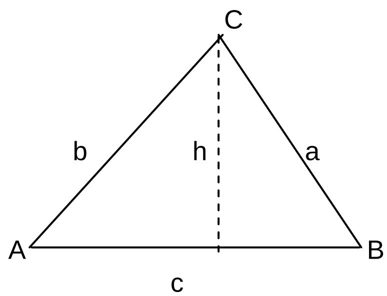


Figure 7.14

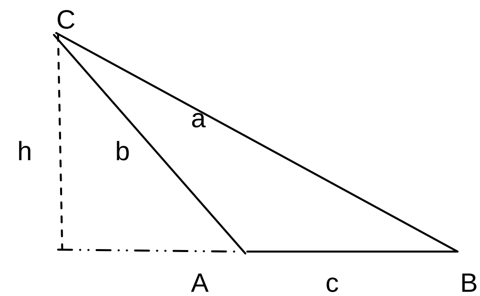


Figure 7.15

In both cases, $\sin A = \frac{h}{b}, \sin B = \frac{h}{a} \Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} (= h)$.

If we draw the perpendiculars from B rather than from C and note $\sin(180^0 - A) = \sin A$, we get $\frac{\sin C}{c} = \frac{\sin A}{a} (= h')$

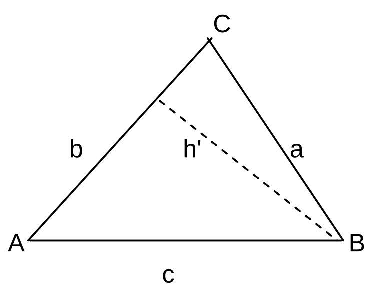


Figure 7.16

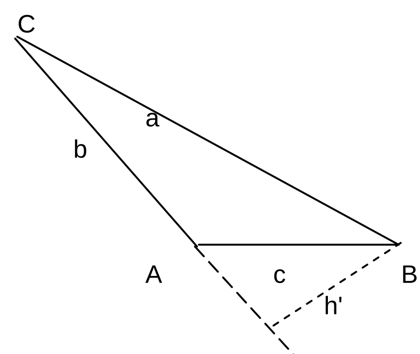


Figure 7.17

The Law of Sines is $\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.}$

Example 10

Solve the given triangle.

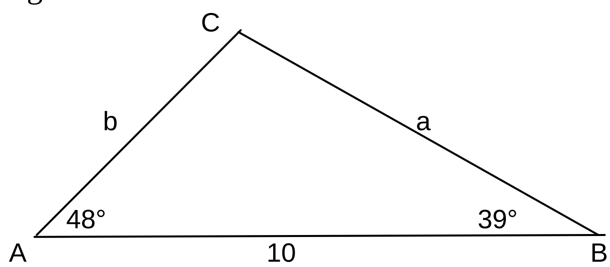


Figure 7.18

<u>Solution</u>

$(i)\ C = 180^0 - 48^0 - 39^0 = 93^0$

$(ii)\ \dfrac{\sin A}{a} = \dfrac{\sin C}{c} \Rightarrow \dfrac{\sin 48^0}{a} = \dfrac{\sin 93^0}{10} \Rightarrow a = 7.4$

$(iii)\ \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \Rightarrow \dfrac{\sin 39^0}{b} = \dfrac{\sin 93^0}{10} \Rightarrow b = 6.3$

Example 11

A surveyor creates the following diagram to measure the width AB of a lake. Find it.

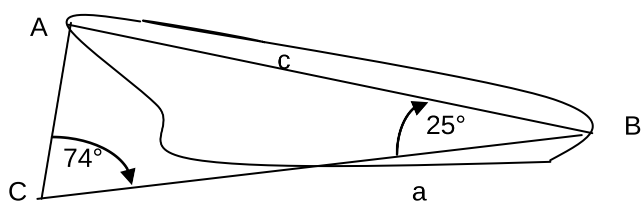


Figure 7.19

<u>Solution</u> $\dfrac{\sin C}{c} = \dfrac{\sin B}{b} \Rightarrow \dfrac{\sin 74^0}{c} = \dfrac{\sin 25}{150} \Rightarrow c = AB = 341.$

The Ambiguous Case

Two angles and one side as in Example 10 above determine a unique triangle, but if two sides and an angle opposite one of them are given, there are three possible situations,

(a) no such triangle exists
(b) one such triangle exists
(c) two distinct triangles satisfy the given conditions – the ambiguous case.

Suppose we are given a, b and A with $h = b\sin A$.

If A is acute, we can have either of the following.

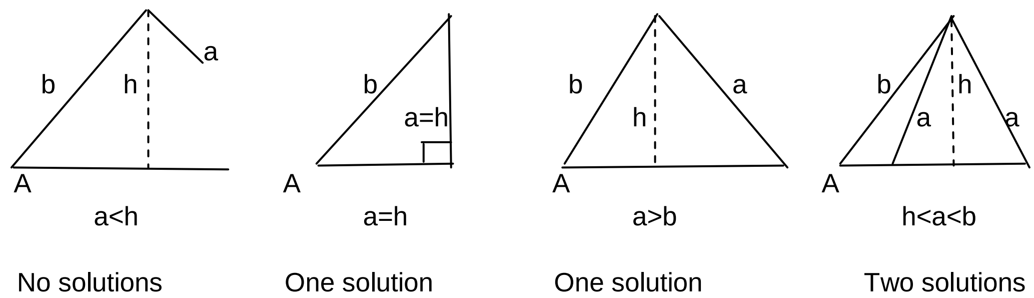


Figure 7.20

If A is obtuse, we can have either of,

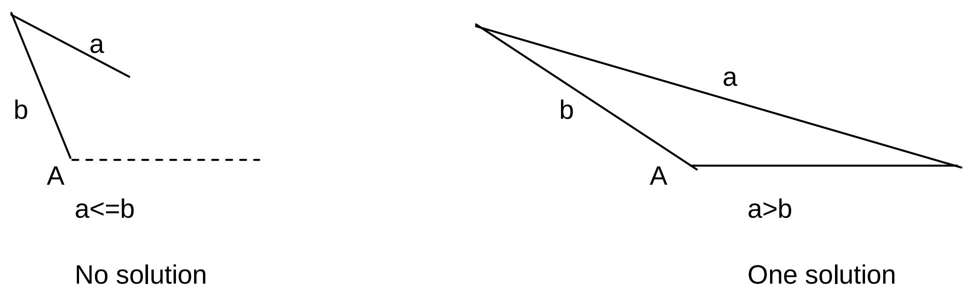


Figure 7.21

Example 12 (Single solution)

Solve the triangle

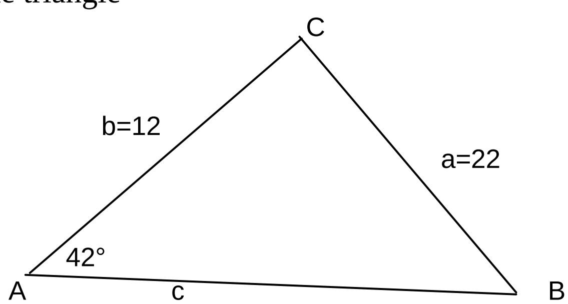


Figure 7.22

<u>Solution</u>

$$(i)\ \frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{12} = \frac{\sin 42^0}{22} \Rightarrow B = 21.41^0$$

$$(ii)\ then\ C = 180^0 - 42^0 - 21.41^0 = 116.59^0$$

$$(iii)\ \frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 116.59^0}{c} = \frac{\sin 42^0}{22} \Rightarrow c = 29.40$$

Example 13 (No solutions)

Attempt to solve the ΔABC for $b = 25, a = 15, A = 85^0$

<u>Solution</u>

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{25} = \frac{\sin 85^0}{15} \Rightarrow \sin B = 1.660 > 1$$

Since we must have $-1 \le \sin B \le 1$, there are no solutions.

Example 14 (Two solutions)

Solve the ΔABC for $a = 12, b = 31, A = 20.5^0$

Solution

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{31} = \frac{\sin 20.5^0}{12} \Rightarrow \sin B = 0.9047$$

$$\Rightarrow B = 64.8^0 \quad or \quad 180^0 - 64.8^0 = 115.2^0$$

$$\Rightarrow C = 180^0 - 64.8^0 - 20.5^0 = 94.7^0$$

OR

$$C = 180^0 - 115.2^0 - 20.5^0 = 44.3^0$$

Using $\frac{\sin C}{c} = \frac{\sin A}{a}$, we get the two solutions, $c = 34.2 \quad or \quad 23.9$

The resulting triangles are,

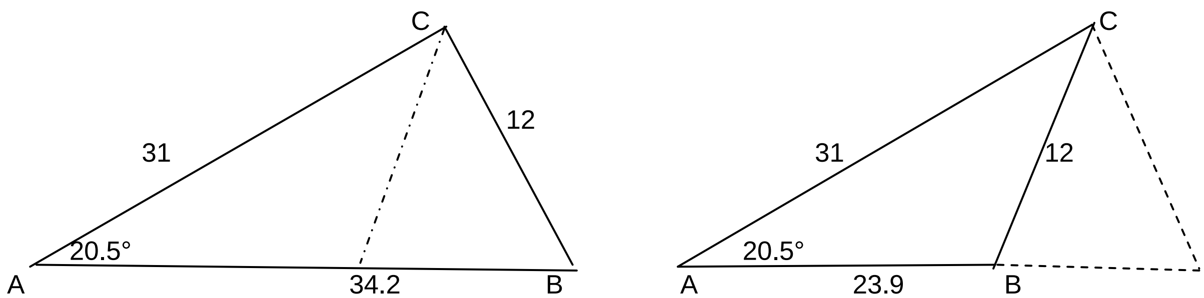


Figure 7.23

Example 15

A rescue ship is 4 miles due west of a crippled destroyer traveling at 10 mph in the direction $N62^0W$. If the rescue ship travels at 18 mph, in what direction should it travel in order to intercept the destroyer?

Solution If the rescue ship is at A and the destroyer is at B, we have the situation in the Figure 7.24.

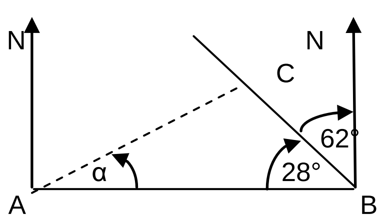


Figure 7.24

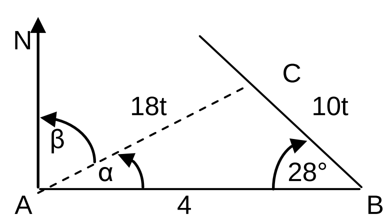


Figure 7.25

Let t be the number of hours before they meet. The rescue ship travels $18t$ and the destroyer travels $10t$ in this time. We then have the picture shown in Figure 7.25. Then,

$$\frac{a}{\sin A}=\frac{b}{\sin B}\Rightarrow\frac{10t}{\sin\alpha}=\frac{18t}{\sin 28^0}\Rightarrow\alpha=15.1^0$$

Answer: $N74.9^0E$

7.5.2 Law of Cosines

We can use the law of cosines to solve triangles where we are given three sides or two sides and the angle enclosed by them. The law of cosines is,

$$a^2=b^2+c^2-2bc\cos A\ \ or,$$
$$b^2=a^2+c^2-2ac\cos B,\ or,$$
$$c^2=a^2+b^2-2ab\cos C$$

<u>Proof</u>

Assume C is an acute angle. Consider the following picture.

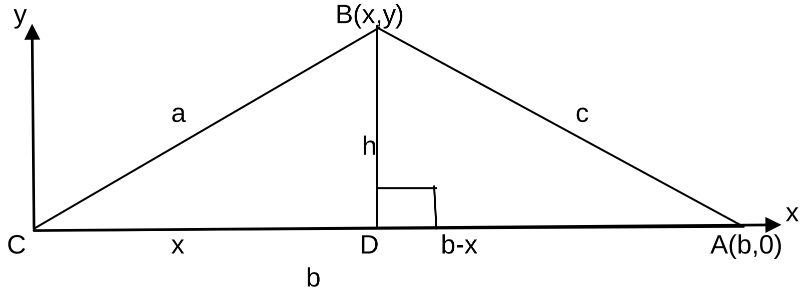


Figure 7.26

Then, $\cos C=\frac{x}{a},\ \sin C=\frac{h}{a}\Rightarrow x=a\cos C,\ h=a\sin C$

Using the Pythagorean Theorem in the ΔDAB, we have

$$\begin{aligned}c^2&=(b-x)^2+h^2\\&=(b-a\cos C)^2+(a\sin C)^2\\&=b^2+a^2(\sin^2C+\cos^2C)-2ab\cos C\\&=a^2+b^2-2ab\cos C\end{aligned}$$

The other two forms and the case of C obtuse are left to the reader.

Example 16

Find a in the triangle shown.

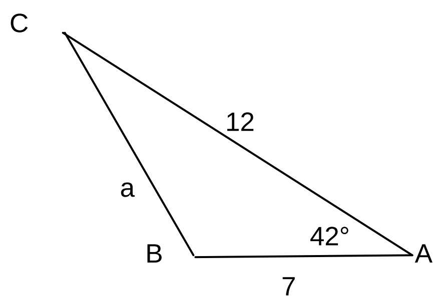


Figure 7.27

Solution

$a^2 = b^2 + c^2 - 2bc\cos A$

$= 49 + 144 - 2 \bullet 7 \bullet 12\cos 42^0$

$\Rightarrow a = 8.3$

Example 17

Find the angle B in the triangle shown.

Solution

$b^2 = a^2 + c^2 - 2ac\cos b$

$19^2 = 8^2 + 14^2 - 2 \bullet 8 \bullet 14\cos B$

$\Rightarrow \cos B = -0.4509$

So B is in the second quadrant,

$B = 180^0 - 53.2^0 = 116.8^0$,

where we used

$\cos^{-1} 0.4509 = 53.2^0$.

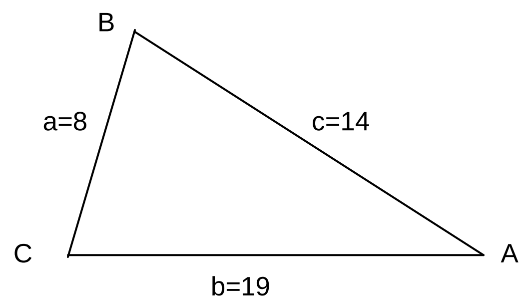


Figure 7.28

Example 18

A ship travels 60 miles due east and then alters course and travels 80 miles in an unknown northerly direction. If it is now 139 miles from its starting point, what was its unknown bearing?

Solution

$b^2 = a^2 + c^2 - 2ac\cos b$

$139^2 = 80^2 + 60^2 - 2 \bullet 80 \bullet 60\cos B$

$\Rightarrow \cos B = -0.97094$

$\Rightarrow B = 166.15^0$

We have the picture shown right.

Answer: $\beta = N\ 76.15^0 E$

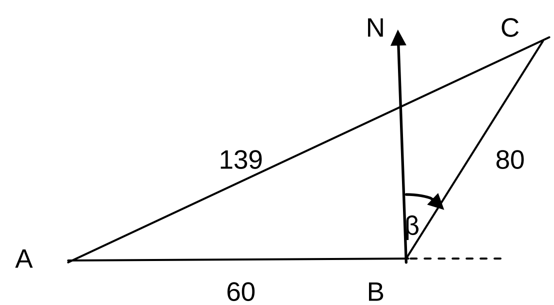


Figure 7.29

Exercises 7B

In 1 through 5, use the Law of Sines to solve the triangles ABC.

1. $A = 35^0, B = 40^0, c = 10$
2. $A = 25^0, B = 35^0, a = 3.5$
3. $C = 105^0, B = 10^0, c = 45$
4. $A = 24.3^0, C = 54.6^0, c = 2.68$
5. $A = 5^0 40', B = 8^0 15', b = 4.8$

6. The Leaning Tower of Pisa in Italy makes an angle of about 84.7^0 with the ground. If you walk 171 feet directly away from the base of the tower so that the tower is leaning directly towards you, the angle of elevation of the top of the tower is 50^0. Find the distance to the nearest foot from the base of the tower to the top of the tower.
7. A pine tree growing vertically upwards on a sloping hillside makes an angle of 75^0 with upwards direction of the hillside. From a point 80 feet up the hill, the angle of elevation of the top of the tree is 62^0 and the angle of depression to the bottom of the tree is 23^0. Find the height of the tree to the nearest foot.
8. You are standing facing the sun with a telephone pole between you and the sun. For you, the angle of elevation of the sun is 62^0. The telephone pole is tilted at an angle of 8^0 to the vertical directly towards you. You find you are actually standing on the tip of the shadow cast by the pole on the ground and you are 20 feet away from the foot of the pole. Find the length of the pole to the nearest foot.
9. At a point A, you find the angle of elevation of the top C of a Redwood tree is 37^0. You walk 100 feet directly towards the tree to point B and find the angle of elevation of the top of the tree is now 44^0. Find the height of the tree to the nearest foot. (Hint: first find BC).

Use the Law of Cosines to solve the triangles ABC in 10 through 12.

10. $a = 7, b = 3, c = 8$
11. $b = 15, c = 30, A = 30^0$
12. $a = 1.42, b = 0.75, c = 1.25$

Determine whether the Law of Sines or the Law of Cosines is needed to solve the triangle ABC in each of 13 through 15. Then solve the triangle.

13. $a = 10, b = 12, C = 70^0$
14. $a = 11, b = 13, c = 7$
15. $a = 160, B = 12^0, C = 7^0$

16. Two ships leave a harbor at the same time. One ship travels on a bearing of $S12^0W$ at 14 mph. The other ship travels on a bearing of $N75^0E$ at 10 mph. How far apart to the nearest 10^{th} of a mile will they be after three hours?
17. Your boat travels 25 miles east from its berth. You then turn and travel 13.5 miles on the bearing $S40^0W$.
    a. How far are you from the berth?
    b. What bearing could you have taken from the berth to reach this spot directly?
18. A baseball diamond is a square of side 90 feet. The pitcher's mound is 60.5 feet from home plate on a line joining second base and home plate. Find the distance from the pitcher's mound to third base (to the nearest 10^{th} of a foot).

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Chapter 8

Trigonometric Identities

8.1 Fundamental Trigonometric Identities

The fundamental trigonometric identities are as follows.

8.1.1 Reciprocal Identities

By definition,

$$\csc x = \frac{1}{\sin x} \Leftrightarrow \sin x = \frac{1}{\csc x} \qquad (1)$$

$$\sec x = \frac{1}{\cos x} \Leftrightarrow \cos x = \frac{1}{\sec x} \qquad (2)$$

$$\cot x = \frac{1}{\tan x} \Leftrightarrow \tan x = \frac{1}{\cot x} \qquad (3)$$

8.1.2 Tangent and Cotangent Identities

By definition,

$$\tan x = \frac{\sin x}{\cos x} \qquad (4)$$

$$\cot x = \frac{\cos x}{\sin x} \qquad (5)$$

8.1.3 Pythagorean Identities

From the unit circle $x^2 + y^2 = 1$ definition of sine and cosine as $\cos\theta = x,\ \sin\theta = y$, we have

$$\sin^2\theta + \cos^2\theta = 1 \qquad (6)$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad (7) \quad [(6) \div \cos^2\theta]$$

$$1 + \cot^2\theta = \csc^2\theta \qquad (8) \quad [(6) \div \sin^2\theta]$$

With x as the variable we have the alternate forms,

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \Rightarrow \cos^2 x = 1 - \sin^2 x \qquad (9\,\&10)$$

$$\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1 \Rightarrow \tan^2 x - \sec^2 x = 1 \qquad (11\&12)$$

$$1 + \cot^2 x = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1 \Rightarrow \csc^2 x - \cot^2 x = 1 \qquad (13\,\&14)$$

8.1.4 Co-function Identities

From the triangle we have:

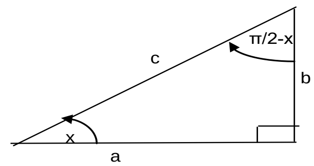


Figure 8.1

$\sin x = \cos(\pi/2 - x)$ (15) *and* $\cos x = \sin(\pi/2 - x)$ (16)

$\tan x = \cot(\pi/2 - x)$ (17) *and* $\cot x = \tan(\pi/2 - x)$ (18)

$\sec x = \csc(\pi/2 - x)$ (19) *and* $\csc x = \sec(\pi/2 - x)$ (20)

We can prove these identities for general $x \in \mathbb{R}$ by positioning the angles θ and $\pi/2-\theta$ in the unit circle.

8.1.5 Unit Circle Reference Angle Identities

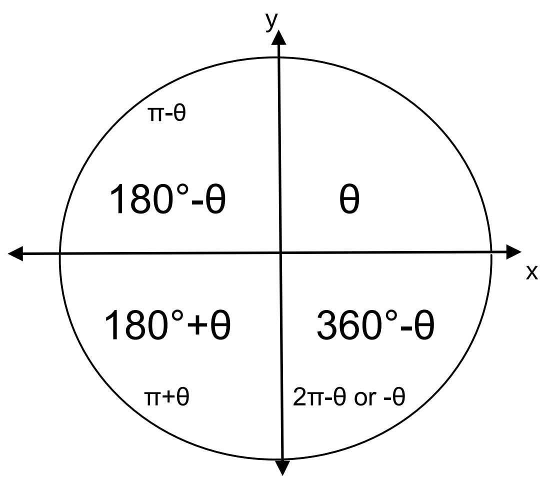


Figure 8.2

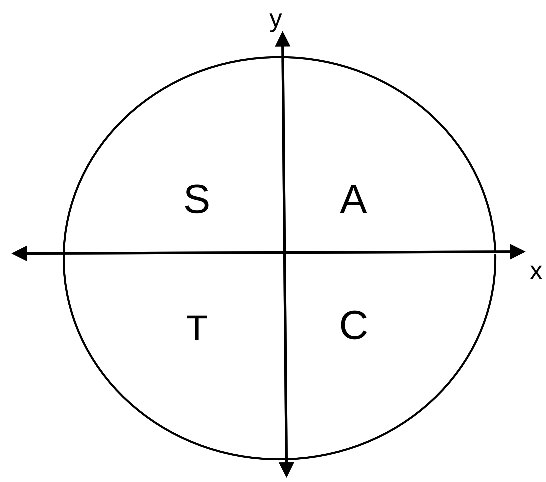


Figure 8.3

$\sin(\pi-\theta) = \sin\theta$ (21) $\cos(\pi-\theta) = -\cos\theta$ (26) $\tan(\pi-\theta) = -\sin\theta$ (30)

$\sin(\pi+\theta) = -\sin\theta$ (23) $\cos(\pi+\theta) = -\cos\theta$ (27) $\tan(\pi+\theta) = \tan\theta$ (31)

$\sin(2\pi-\theta) = -\sin\theta$ (24) $\cos(2\pi-\theta) = \cos\theta$ (28) $\tan(2\pi-\theta) = -\tan\theta$ (32)

$\sin(-\theta) = -\sin\theta$ (25) $\cos(-\theta) = \cos\theta$ (29) $\tan(-\theta) = -\tan\theta$ (33)

For $n \in \mathbb{Z}$,

$\sin(\theta + 2\pi n) = \sin\theta$ (34)

$\cos(\theta + 2\pi n) = \cos\theta$ (35)

$\tan(\theta + \pi n) = \tan\theta$ (36)

We can replace π with 180^0 in each of the identities (21) through (36).

8.2 Finding the Values of Trigonometric Functions

If we are given the value of any trigonometric function, we can use a right angle triangle containing the reference angle to find the values of the other trigonometric functions.

Example 1

Use $\sec x = -\frac{3}{2}$ and $\tan x > 0$ to find the values of all six trigonometric functions.

<u>Solution</u> $\sec x = -\frac{3}{2} \Rightarrow \cos x = -\frac{2}{3}$

The reference angle x' is as shown in the right triangle, Figure 8.4. Since cos is negative and tan is positive, ASTC tells us x is in the third quadrant or $x = \pi + x'$.

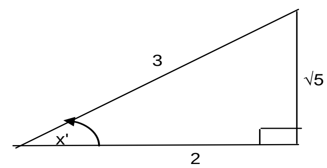


Figure 8.4

$$\sin x = \sin(\pi + x') = -\sin x' = -\frac{\sqrt{5}}{3}, \quad \csc x = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\cos x = -\frac{2}{3}, \quad \sec x = -\frac{3}{2}$$

$$\tan x = \tan(\pi + x') = \tan x' = \frac{\sqrt{5}}{2}, \quad \cot x = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

8.3 Simplifying Trigonometric Expressions

We can use the fundamental identities to simplify and manipulate trigonometric expressions.

Example 2

Simplify $\sin x \cos^2 x - \sin x$

<u>Solution</u>

$$\begin{aligned} &\sin x \cos^2 x - \sin x \\ &= -\sin x(1 - \cos^2 x) \\ &= -\sin x(\sin^2 x) \\ &= -\sin^3 x \end{aligned}$$

Example 3

Factor (a) $\tan^2 x - 1$ (b) $4\sin^2 x + \sin x - 3$

<u>Solution</u>

$(a)\ \tan^2 x - 1 = (\tan x + 1)(\tan x - 1)$

$(b)\ 4\sin^2 x + \sin x - 3 = (4\sin x - 3)(\sin x + 1)$

Example 4
Simplify $\sin\theta + \cot\theta\cos\theta$

<u>Solution</u> $\sin\theta + \cot\theta\cos\theta = \sin\theta + \dfrac{\cos\theta}{\sin\theta}\cos\theta = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta} = \dfrac{1}{\sin\theta} = \csc\theta$

Example 5
Simplify $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

<u>Solution</u> $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} = \dfrac{\cancel{1+\cos\theta}}{\sin\theta(\cancel{1+\cos\theta})} = \csc\theta$

Example 6
Simplify $\tan x + \dfrac{\cos x}{1+\sin x}$

<u>Solution</u>

$$\tan x + \frac{\cos x}{1+\sin x} = \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x} = \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)} = \frac{\cancel{1+\sin x}}{\cos x\,(\cancel{1+\sin x})} = \sec x$$

Example 7
Simplify $\sec^2 x - \tan x \sec x$

<u>Solution</u>

$$\begin{aligned}
\sec^2 x - \tan x \sec x &= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\frac{1}{\cos x} \\
&= \frac{1-\sin x}{\cos^2 x} \\
&= \frac{1-\sin x}{1-\sin^2 x} \\
&= \frac{\cancel{1-\sin x}}{(\cancel{1-\sin x})\,(1+\sin x)} \\
&= \frac{1}{1+\sin x}
\end{aligned}$$

Exercises 8A

Find the exact value of the trigonometric functions in exercises 1 through 5.

1. $(a)\sin\frac{2\pi}{3}$ $(b)\cos\frac{2\pi}{3}$ $(c)\tan\frac{2\pi}{3}$
2. $(a)\sin\frac{7\pi}{6}$ $(b)\sin-\frac{\pi}{6}$ $(c)\sin\frac{11\pi}{6}$
3. $(a)\cos\left(-\frac{\pi}{3}\right)$ $(b)\sec\left(-\frac{\pi}{3}\right)$ $(c)\tan\left(-\frac{\pi}{3}\right)$
4. $(a)\sin\left(-\frac{3\pi}{2}\right)$ $(b)\cos\left(-\frac{3\pi}{2}\right)$ $(c)\cot\left(-\frac{3\pi}{2}\right)$
5. $(a)\tan\frac{5\pi}{6}$ $(b)\tan\frac{7\pi}{6}$ $(c)\tan\frac{11\pi}{6}$

In exercises 6 through 11, find the values of the 6 trigonometric functions of θ from the given information

6. $\sin\theta = \frac{3}{5}$, terminal point of θ is in Quadrant II.
7. $\cos\theta = -\frac{4}{5}$, terminal point of θ is in Quadrant III.
8. $\sec\theta = 3$, terminal point of θ is in Quadrant IV.
9. $\tan\theta = \frac{1}{4}$, terminal point of θ is in Quadrant III.
10. $\sec\theta = 2$, $\sin\theta < 0$.
11. $\sin\theta = -\frac{1}{4}$, $\sec\theta < 0$.

Write the expression in problems 12 to 25 in terms of sine and cosine and then simplify.

12. $\cos\theta\tan\theta$
13. $\cos\theta\csc\theta$
14. $\sin\theta\sec\theta$
15. $\tan^2 x - \sec^2 x$
16. $\frac{\sec x}{\csc x}$
17. $\cot x \sin x$
18. $\sin\phi(\csc\phi - \sin\phi)$
19. $\frac{\csc x}{\cot x}$
20. $\sec\alpha \bullet \frac{\sin\alpha}{\tan\alpha}$
21. $\sin(\pi/2 - x)\csc x$
22. $\sin u + \cot u \cos u$
23. $\cos^2\theta(1+\tan^2\theta)$
24. $\frac{\sec\theta - \cos\theta}{\sin\theta}$
25. $\frac{\cot\theta}{\csc\theta - \sin\theta}$

Simplify the trigonometric expressions 26 to 49.

26. $\frac{\sin x \sec x}{\tan x}$
27. $\frac{\cos^2 y}{1-\sin y}$
28. $\sin\theta + \cos\theta\cot\theta$
29. $\frac{\cos\theta}{1-\sin\theta}$

30. $\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}$

31. $\dfrac{\sin\theta+\cos\theta}{\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta}$

32. $\cos^3 x+\sin^2 x\cos x$

33. $\dfrac{1+\cos y}{1+\sec y}$

34. $\dfrac{\tan x}{\sec(-x)}$

35. $\dfrac{\sec^2 x-1}{\sec^2 x}$

36. $\dfrac{\sec x-\cos x}{\tan x}$

37. $\sin\theta\csc\theta-\sin^2\theta$

38. $1-\dfrac{\sin^2\theta}{1-\cos\theta}$

39. $\dfrac{\tan\theta}{1+\sec\theta}+\dfrac{1+\sec\theta}{\tan\theta}$

40. $(\sin x+\cos x)^2$

41. $\dfrac{1}{1+\cos x}+\dfrac{1}{1-\cos x}$

42. $\dfrac{1+\csc x}{\cos x+\cot x}$

43. $\dfrac{\sin x}{\csc x}+\dfrac{\cos x}{\sec x}$

44. $\dfrac{1+\sin u}{\cos u}+\dfrac{\cos u}{1+\sin u}$

45. $\tan x\cos x\csc x$

46. $\dfrac{2+\tan^2 x}{\sec^2 x}-1$

47. $\dfrac{1+\cot A}{\csc A}$

48. $\tan\theta+\cos(-\theta)+\tan(-\theta)$

49. $\dfrac{\cos x}{\sec x+\tan x}$

8.4 Verifying Trigonometric Identities

An identity is verified if we show the two sides are the same, that is, simplify to the same result.

Example 8

Show $\tan\theta+\cot\theta=\sec^2\theta\cot\theta$

Solution

LeftSide or LS

$$=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}$$

RightSide or RS

$$=\sec^2\theta\cot\theta$$

$$=\frac{1}{\cos^2\theta}\frac{\cos\theta}{\sin\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}$$

$$=LS$$

Example 9

Show $(\sin x-\cos x)(\csc x+\sec x)=\tan x-\cot x$

Solution

LS

$$=(\sin x-\cos x)(\frac{1}{\sin x}+\frac{1}{\cos x})$$

$$=\not{1}+\frac{\sin x}{\cos x}-\frac{\cos}{\sin x}-\not{1}$$

$$=\tan x-\cot x$$

$$=RS$$

Example 10

Show $\dfrac{\sec\theta-\cos\theta}{\csc\theta-\sin\theta}=\tan^3\theta$

Solution

$$LS=\frac{\frac{1}{\cos\theta}-\cos\theta}{\frac{1}{\sin\theta}-\sin\theta}=\frac{1-\cos^2\theta}{\cos\theta}\div\frac{1-\sin^2\theta}{\sin\theta}$$

$$=\frac{\sin^2\theta}{\cos\theta}\bullet\frac{\sin\theta}{\cos^2\theta}=\tan^3\theta=RS$$

Example 11

Show $\dfrac{\sec^2 x - 1}{\sec^2 x} = \sin^2 x$

<u>Solution</u> $LS = \dfrac{sec^2 x - 1}{\sec^2 x} = \tan^2 x \bullet \cos^2 x = \dfrac{\sin^2 x}{\cancel{\cos^2 x}} \bullet \cancel{\cos^2 x} = RS$

Example 12

Show $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

<u>Solution</u>

$$LS = \frac{1}{1-\sin x} + \frac{1}{1+\sin x} = \frac{1+\cancel{\sin x} + 1 \cancel{-\sin x}}{(1-\sin x)(1+\sin x)}$$

$$= \frac{2}{1-\sin^2 x} = \frac{2}{\cos^2 x} = 2\sec^2 x = RS$$

Example 13

Show $\dfrac{\cot^2 x}{1+\csc x} = \dfrac{1-\sin x}{\sin x}$

<u>Solution</u>

$$LS = \frac{\cot^2 x}{1+\csc x} = \frac{\frac{\cos^2 x}{\sin^2 x}}{1+\frac{1}{\sin x}} = \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1+\sin x}{\sin x}} = \frac{\cos^2 x}{\sin^{\cancel{2}} x} \bullet \frac{\cancel{\sin x}}{1+\sin x}$$

$$= \frac{1-\sin^2 x}{\sin x(1+\sin x)} = \frac{(1-\sin x)\,\cancel{(1+\sin x)}}{\sin x\,\cancel{(1+\sin x)}} = \frac{1-\sin x}{\sin x} = RS$$

Example 14

Show $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$

<u>Solution</u> $RS = \tan^2 x(\sec^2 x - 1) = \tan^2 x \tan^2 x = \tan^4 x = LS$

Exercises 8B

Verify the following identities.

1. $1+\cot^2(-\theta)=\csc^2\theta$
2. $\dfrac{\sin\theta}{\tan\theta}=\cos\theta$
3. $\sin x(\cot x+\tan x)=\sec x$
4. $\dfrac{\cos u\sec u}{\tan u}=\cot u$
5. $\sin u\csc u-\cos^2 u=\sin^2 u$
6. $\sin B+\cos B\cot B=\csc B$
7. $(\csc x-1)(\csc x+1)=\cot^2 x$
8. $\cot(-\alpha)\cos(-\alpha)+\sin(-\alpha)=-\csc\alpha$
9. $(1-\cos^2 x)((1+\cot^2 x)=1$
10. $\tan\theta+\cot\theta=\sec\theta\csc\theta$
11. $(1-\cos\beta)(1+\cos\beta)=\dfrac{1}{\csc^2\beta}$
12. $\dfrac{1+\tan u}{1-\tan u}=\dfrac{\cot u+1}{\cot u-1}$
13. $\dfrac{(\sin x+\cos x)^2}{\sin^2 x-\cos^2 x}=\dfrac{\sin^2 x-\cos^2 x}{(\sin x-\cos x)^2}$
14. $\dfrac{\sec t-\cos t}{\sec t}=\sin^2 t$
15. $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$
16. $\dfrac{1}{1-\sin^2 y}=1+\tan^2 y$
17. $(\cot x-\csc x)(\cos x+1)=-\sin x$
18. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\sin^2\theta-\cos^2\theta$
19. $(1-\cos^2 x)(1+\cot^2 x)=1$
20. $2\cos^2 x-1=1-2\sin^2 x$
21. $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}+1=2\cos^2\theta$
22. $\dfrac{1-\cos\alpha}{\sin\alpha}=\dfrac{\sin\alpha}{1+\cos\alpha}$
23. $\tan^2\theta-\sin^2\theta=\tan^2\theta\sin^2\theta$
24. $\dfrac{\sin x-1}{\sin x+1}=\dfrac{-\cos^2 x}{(\sin x+1)^2}$
25. $\sec x-\cos x=\sin x\tan x$
26. $\dfrac{(\sin t+\cos t)^2}{\sin t\cos t}=2+\sec t\csc t$
27. $\dfrac{1+\tan^2 u}{1-\tan^2 u}=\dfrac{1}{\cos^2 u-\sin^2 u}$
28. $\dfrac{\sec x}{\sec x-\tan x}=\sec x(\sec x+\tan x)$
29. $\sec v-\tan v=\dfrac{1}{\sec v+\tan v}$
30. $\dfrac{\sin x+\cos x}{\sec x+\csc x}=\sin x\cos x$
31. $\dfrac{\csc x-\cot x}{\sec x-1}=\cot x$
32. $\tan^2 u-\sin^2 u=\tan^2 u\sin^2 u$
33. $\dfrac{(\sec u-\tan u)^2+1}{\csc u(\sec u-\tan u)}=2\tan u$
34. $\dfrac{1-2\cos^2 x}{\sin x\cos x}=\tan x-\cot x$
35. $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$
36. $\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{\sin\theta-\csc\theta}{\cos\theta-\cot\theta}$
37. $\dfrac{\cos^2 t+\tan^2 t-1}{\sin^2 t}=\tan^2 t$
38. $\dfrac{1}{\sec x+\tan x}+\dfrac{1}{\sec x-\tan x}=2\sec x$
39. $(\tan x+\cot x)^2=\sec^2 x+\csc^2 x$
40. $\dfrac{\cos^2 u-\sin^2 u}{1-\tan^2 u}=\cos^2 u$
41. $\dfrac{\sec u-1}{\sec u+1}=\dfrac{1-\cos u}{1+\cos u}$
42. $\dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}=1-\sin x\cos x$
43. $\dfrac{1+\sin x}{1-\sin x}=(\tan x+\sec x)^2$

8.5 Solving Trigonometric Equations

Recall

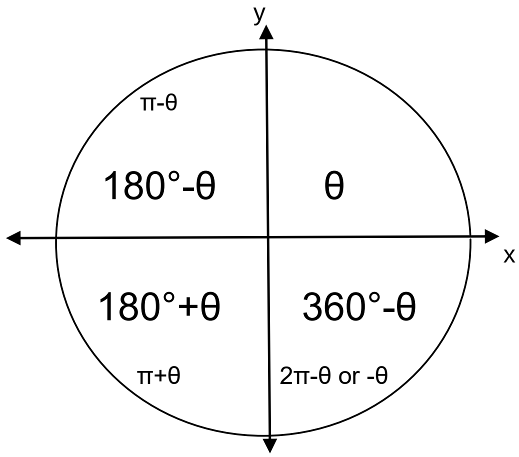


Figure 8.5

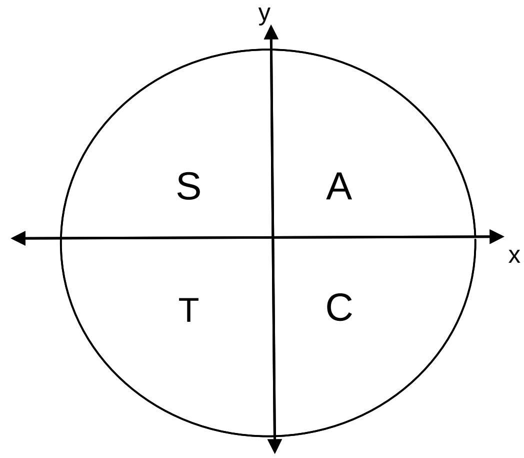


Figure 8.6

For $n \in \mathbb{Z}$,

$$\sin(\theta + 2\pi n) = \sin\theta, \quad \cos(\theta + 2\pi n) = \cos\theta, \quad \tan(\theta + 2\pi n) = \tan\theta$$

Example 15

Solve $2\sin x - 1 = 0$ for $(a)\ x \in [0, 2\pi],\ (b)\ x \in \mathbb{R}$

<u>Solution</u>

$$2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2}$$

$$(a)\ x = \frac{\pi}{6}, \pi - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$(b)\ x = \frac{\pi}{6} + 2\pi n,\ \frac{5\pi}{6} + 2\pi n,\ n \in \mathbb{Z}$$

Example 16

Solve $\tan^2\theta - 1 = 0,\ \theta \in \mathbb{R}$

<u>Solution</u> $\tan\theta = \pm 1$, so the answers are in any of the four quadrants.

$$\theta = \frac{\pi}{4} + n\pi,\ \pi - \frac{\pi}{4} + n\pi,\ \pi + \frac{\pi}{4} + n\pi,\ 2\pi - \frac{\pi}{4} + n\pi$$

$$= \frac{\pi}{4} + n\pi,\ \frac{3\pi}{4} + n\pi,\ \frac{5\pi}{4} + n\pi,\ \frac{7\pi}{4} + n\pi$$

$$= \frac{\pi}{4} + n\frac{\pi}{2}$$

(Convince yourself this is true!)

Example 17

Solve $\sin^2 x - 3\sin x + 2 = 0,\ 0^0 \le \theta \le 360^0$

<u>Solution</u> $(\sin x - 2)(\sin x - 1) = 0 \Rightarrow \sin x = 1, 2$

But $\sin x = 2$ is impossible. (Why?)

The only solution is $\sin x = 1 \Rightarrow x = 90^0$

Example 18

Solve $\sin^2 x - \cos^2 x = 1,\ x \in [0, 2\pi]$

<u>Solution</u>

$$\sin^2 x - \cos^2 x = 1$$

$$\Leftrightarrow \sin^2 x - (1 - \sin^2 x) = 1$$

$$\Leftrightarrow 2\sin^2 x = 2$$

$$\Rightarrow \sin x = \pm 1$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Exercises 8C

Solve the given equation for $\theta \in [0, 2\pi]$.

1. $\sin x = \frac{1}{2}$
2. $\cos x = -\frac{\sqrt{3}}{2}$
3. $\sin x = 1$
4. $\tan x + 1 = 0$
5. $\csc x = 1$
6. $\tan \theta = -\sqrt{3}$
7. $3\cos x + 1 = 5$
8. $\sin \theta = \frac{\sqrt{3}}{2}$
9. $\cos \theta = -1$
10. $\cos \theta = \frac{\sqrt{3}}{2}$
11. $\tan \theta = 1$
12. $\sin \theta = \frac{\sqrt{2}}{2}$
13. $\cos \theta + 1 = 0$
14. $5\sin \theta - 1 = 0$
15. $\sin^2 x + 4 = 5$

16. $4\cos^2 x - 3 = 0$
17. $6\cot^2\theta + 1 = 3$
18. $\cos^2 x + 2 = 3\cos x$
19. $\tan^2\theta = \tan\theta$
20. $2\sin x = \sin x \cos x$
21. $\sin x = \cos x$
22. $2\cos^2\theta - 1 = 0$
23. $\tan^2\theta - 4 = 0$
24. $\sec^2\theta - 2 = 0$
25. $(\tan^2\theta - 4)(2\cos\theta + 1) = 0$
26. $3\sin^2\theta - 7\sin\theta + 2 = 0$
27. $2\cos^2\theta - 7\cos\theta + 3 = 0$
28. $\cos^2\theta - \cos\theta - 6 = 0$
29. $\cot^2 u - \csc u = 1$
30. $\sin\theta + \cos\theta = 1$ (hint, square both sides)
31. $\tan x + 1 = \sec x$ (hint: square both sides)

Solve for $x \in \mathbb{R}$

32. $\sin^2\theta = 2\sin\theta + 3$
33. $\cos\theta(2\sin\theta + 1) = 0$
34. $\cos\theta\sin\theta - 2\cos\theta = 0$
35. $3\tan\theta\sin\theta - 2\tan\theta = 0$
36. $2\cos^2\theta + \sin\theta = 1$
37. $\sin^2\theta = 4 - 2\cos^2\theta$
38. $\tan^2\theta - 2\sec\theta = 2$
39. $\csc^2\theta = \cot\theta + 3$
40. $2\sin 2\theta - 3\sin\theta = 0$
41. $3\sin 2\theta - 2\sin\theta = 0$
42. $\cos 2\theta = 3\sin\theta - 1$
43. $\cos 2\theta = \cos^2\theta - \frac{1}{2}$
44. $2\sin^2\theta - \cos\theta = 1$
45. $\tan\theta - 3\cot\theta = 0$
46. $\sin\theta - 1 = \cos\theta$
47. $\cos\theta - \sin\theta = 1$
48. $\tan\theta + 1 = \sec\theta$
49. $2\tan\theta + \sec^2\theta = 4$
50. $2\cos 3\theta = 1$
51. $3\csc^2\theta = 4$
52. $2\cos 2\theta + 1 = 0$
53. $2\sin 3\theta + 1 = 0$

=====================

8.6 Addition Formulas

If A, B are any two angles, we prove the Addition formulas,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad (37)$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \qquad (38)$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad (39)$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad (40)$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad (41)$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \qquad (42)$$

<u>Proof</u>

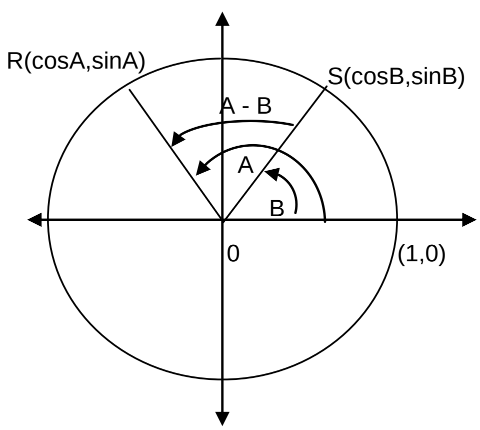


Figure 8.7

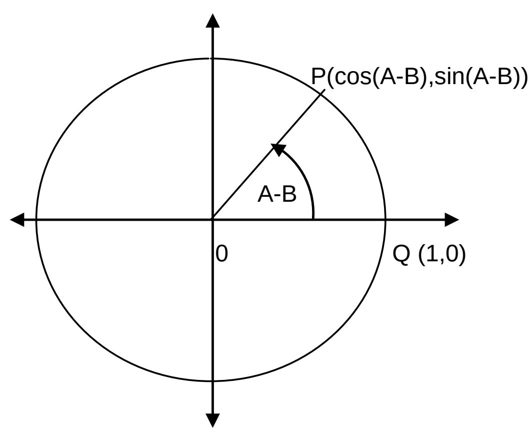


Figure 8.8

We obtain Figure 8.8 from Figure 8.7 by repositioning the arc RS so that S is at $(1,0)$.

Clearly $PQ = RS \Rightarrow PQ^2 = RS^2$

Using the distance formula,

$$(\cos(A-B)-1)^2 + \sin^2(A-B) = [\cos A - \cos B]^2 + [\sin A - \sin B]^2$$
$$\Rightarrow \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B)$$
$$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$$
$$\Rightarrow -2\cos(A-B) + \not{2} = -2\cos A\cos B - 2\sin A\sin B + \not{2}$$
$$\Rightarrow \cos(A-B) = \cos A\cos B + \sin A\sin B \qquad (40)$$

Putting $B = -B$ and using $\sin(-x) = -\sin x,\ \cos(-x) = \cos x$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad (39)$$

Using $\sin x = \cos(\pi/2 - x),\ \mathit{and}\ \cos x = \sin(\pi/2 - x)$, we get,

$$\sin(A+B)=\cos(\pi/2-(A+B)=\cos([\pi/2-A]+B)$$
$$=\cos(\pi/2-A)\cos B+\sin(\pi/2-A)\sin B$$
$$=\sin A\cos B+\cos A\sin B \qquad (37)$$

Put $B=-B$ to get
$$\sin(A-B)=\sin A\cos B-\cos A\sin B \qquad (38)$$

Finally,
$$\tan(A\pm B)=\frac{\sin(A\pm B)}{\cos(A\pm B)}=\frac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B\mp\sin A\sin B}$$
$$=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \qquad (41)\,\&\,(42)$$

on dividing numerator and denominator by $\cos A\cos B$.

Example 19

Find the exact values of $(a)\sin\frac{\pi}{12}$ $(b)\cos 75^0$ $(c)\tan 15^0$

Solution

$$(a)\sin\frac{\pi}{12}=\sin(\frac{\pi}{3}-\frac{\pi}{4})=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\bullet\frac{\sqrt{2}}{2}-\frac{1}{2}\bullet\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$(b)\cos 75^0=\cos(45^0+30^0)=\cos 45^0\cos 30^0-\sin 45^0\sin 30^0=\frac{\sqrt{2}}{2}\bullet\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\bullet\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$(c)\tan 15^0=\tan(45^0-30^0)=\frac{\tan 45^0-\tan 30^0}{1+\tan 45^0\tan 30^0}=\frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}=\frac{3-\sqrt{3}}{3+\sqrt{3}}\bullet\frac{3-\sqrt{3}}{3-\sqrt{3}}=2-\sqrt{3}$$

Example 20

Find the exact value of $\sin(\theta-\phi)$ given $\sin\theta=\frac{3}{4}$, $\cos\phi=-\frac{2}{5}$, θ is in the second quadrant and ϕ is in the third quadrant.

Solutions

Use these two triangles. Note θ', ϕ' are the reference angles.

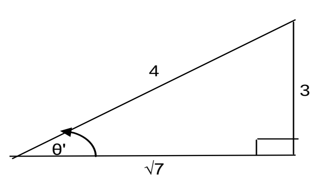


Figure 8.9

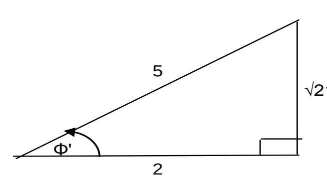


Figure 8.10

Then,

$$\cos\theta = -\frac{\sqrt{7}}{4}, \; \sin\phi = -\frac{\sqrt{21}}{5}$$

$$\begin{aligned}\sin(\theta-\phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi \\ &= \left(\frac{3}{4}\right)\left(-\frac{2}{5}\right) - \left(-\frac{\sqrt{7}}{4}\right)\left(-\frac{\sqrt{21}}{5}\right) \\ &= -\left(\frac{6+7\sqrt{3}}{20}\right)\end{aligned}$$

Example 21

Simplify $(a)\cos(\theta - \frac{3\pi}{4}),\ (b)\ \tan(\theta+3\pi)$

<u>Solution</u>

$$(a)\cos(\theta-\frac{3\pi}{4}) = \cos\theta\cos\frac{3\pi}{4} + \sin\theta\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = \frac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$$

$$(b)\tan(\theta+3\pi) = \frac{\tan\theta + \tan 3\pi}{1-\tan\theta\tan 3\pi} = \tan\theta$$

Exercises 8D

Use an Addition or Subtraction formula to find the exact value of each expression.

1. $\sin 5\pi/12$
2. $\sin \pi/12$
3. $\cos 7\pi/12$
4. $\cos 195^0$
5. $\tan 15^0$
6. $\tan 165^0$
7. $\sin\dfrac{19\pi}{12}$
8. $\cos\dfrac{17\pi}{12}$
9. $\tan\left(-\dfrac{\pi}{12}\right)$
10. $\sin\left(-\dfrac{5\pi}{12}\right)$
11. $\cos\dfrac{11\pi}{12}$
12. $\tan\dfrac{7\pi}{12}$

Use an Addition or Subtraction formula to write the expression as a trigonometric function of one number and then find its exact value.

13. $\sin 20^0 \cos 10^0 + \cos 20^0 \sin 10^0$
14. $\sin 20^0 \cos 80^0 - \cos 20^0 \sin 80^0$
15. $\cos\dfrac{3\pi}{7}\cos\dfrac{2\pi}{21} + \sin\dfrac{3\pi}{7}\sin\dfrac{2\pi}{21}$
16. $\dfrac{\tan\dfrac{\pi}{18} + \tan\dfrac{\pi}{9}}{1 - \tan\dfrac{\pi}{18}\tan\dfrac{\pi}{9}}$
17. $\dfrac{\tan 40^0 - \tan 10^0}{1 + \tan 40^0 \tan 10^0}$
18. $\cos\dfrac{13\pi}{15}\cos\left(-\dfrac{\pi}{5}\right) - \sin\dfrac{13\pi}{15}\sin\left(-\dfrac{\pi}{5}\right)$
19. $\cos 70^0 \cos 20^0 - \sin 70^0 \sin 20^0$
20. $\dfrac{\tan 20^0 + \tan 25^0}{1 - \tan 20^0 \tan 25^0}$
21. $\cos 45^0 \cos 15^0 + \sin 45^0 \sin 15^0$

22. $\sin(\pi/12)\cos(7\pi/12)-\cos(\pi/12)\sin(7\pi/12)$

Prove the following co-function identities using the Addition and Subtraction formulas.

23. $\tan\left(\frac{\pi}{2}-u\right)=\cot u$

24. $\cot\left(\frac{\pi}{2}-u\right)=\tan u$

25. $\sec\left(\frac{\pi}{2}-u\right)=\csc u$

26. $\csc\left(\frac{\pi}{2}-u\right)=\sec u$

Prove the following identities.

27. $\sin(x-\frac{\pi}{2})=-\cos x$

28. $\cos\left(x-\frac{\pi}{2}\right)=\sin x$

29. $\sin(x-\pi)=-\sin x$

30. $\cos(x-\pi)=-\cos x$

31. $\tan(x-\pi)=\tan x$

32. $\sin\left(\frac{\pi}{2}-x\right)=\sin\left(\frac{\pi}{2}+x\right)$

33. $\cos\left(x+\frac{\pi}{6}\right)+\sin\left(x-\frac{\pi}{3}\right)=0$

34. $\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x+1}{\tan x-1}$

35. $\sin(x+y)-\sin(x-y)=2\cos x\sin y$

36. $\cos(x+y)+\cos(x-y)=2\cos x\cos y$

Evaluate each expression under the given conditions.

37. $\cos(\theta-\phi)$; $\cos\theta=\frac{3}{5}$, θ in Quadrant IV, $\tan\phi=-\sqrt{3}$, ϕ in Quadrant II.

38. $\sin(\theta-\phi)$; $\tan\theta=\frac{4}{3}$, θ in Quadrant III, $\sin\phi=-\frac{\sqrt{10}}{10}$, ϕ in Quadrant IV.

39. $\sin(\theta+\phi)$; $\sin\theta=\frac{5}{13}$, θ in Quadrant I, $\cos\phi=-\frac{2\sqrt{5}}{5}$, ϕ in Quadrant II.

40. $\tan(\theta+\phi)$; $\cos\theta=-\frac{1}{3}$, θ in Quadrant III, $\sin\phi=\frac{1}{4}$, ϕ in Quadrant II.

8.7. Double Angle Formulas

If we put $A = x,\ B = x$ in the Addition Formulas we get the double angle formulas,

$$\sin 2x = 2\sin x\cos x \qquad (43)$$

$$\cos 2x = \cos^2 x - \sin^2 x \quad (44)$$

$$= 2\cos^2 x - 1 \qquad (45) \quad (put \ \ \sin^2 x = 1 - \cos^2 x)$$

$$= 1 - 2\sin^2 x \qquad (46) \quad (put \ \ \cos^2 x = 1 - \sin^2 x)$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \qquad (47)$$

Example 22

If $\cos\theta = -\dfrac{5}{13}$ and θ is in the second quadrant, find $(a)\sin 2\theta,\ (b)\ \cos 2\theta$

Solution

Use this triangle. Note θ' is the reference angle.

$$\sin\theta = \sin\theta' = \frac{12}{13}$$

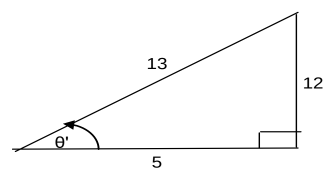


Figure 8.11

$$(a)\sin 2\theta = 2\sin\theta\cos\theta = 2\bullet\left(\frac{12}{13}\right)\bullet\left(-\frac{5}{13}\right) = -\frac{120}{169}$$

$$(b)\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{5}{13}\right)^2 - 1 = -\frac{119}{169}$$

Example 23

Write $\sin 3x$ in terms of $\sin x$.

Solution

$$\sin 3x = \sin(2x + x)$$

$$= \sin 2x\cos x + \cos 2x\sin x$$

$$= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x$$

$$= 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x)\sin x$$

$$= 3\sin x - 4\sin^3 x$$

Exercises 8E

1. Given $\cos x = -\frac{12}{13}$, find $\cos 2x$
2. Given $\tan x = \frac{3}{5}$, find $\tan 2x$
3. Given $\sin x = -\frac{24}{25}$ and x is in the fourth quadrant, find $\sin 2x$
4. Given $\tan x = -\frac{1}{3}$ and x is in the second quadrant, find $\sin 2x$
5. Given $\tan x = -\frac{4}{3}$, x is in the second quadrant, find $\sin 2x, \cos 2x, \tan 2x$
6. Given $\csc x = 4, \tan x < 0$, find $\sin 2x, \cos 2x, \tan 2x$.
7. Given $\sec x = 2$, x in the fourth quadrant, find $\sin 2x, \cos 2x, \tan 2x$
8. Given $\cot x = \frac{2}{3}, \sin x > 0, x$ in Quadrant II, find $\sin 2x, \cos 2x, \tan 2x$.
9. Rewrite $\cos^4 x$ in terms of the first power of cosine.

Prove the following identities.

10. $(\sin x + \cos x)^2 = 1 + \sin 2x$
11. $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$
12. $2\cos^2 x = \cot x \sin 2x$
13. $\sin 8x = 2\sin 4x \cos 4x$
14. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$
15. $\dfrac{1 + \sin 2x}{\sin 2x} = 1 + \frac{1}{2}\sec x \csc x$
16. $\tan x + \cot x = \dfrac{2}{\sin 2x}$
17. $\cot 2x = \dfrac{1 - \tan^2 x}{2\tan x}$
18. $\dfrac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$
19. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$
20. $\tan y = \dfrac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)}$

8.8 Alternative Forms of the Double Angle Formulas

By rearrangement, the double angle formulas (45) and (46) for $\cos 2\theta$ become,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad (48)$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad (49)$$

Example 24
Write $\sin^4 x$ in terms of first degree trigonometric expressions.
Solution

$$\begin{aligned}\sin^4 x &= (\sin^2 x)(\sin^2 x)\\ &= \left(\frac{1-\cos 2x}{2}\right)^2\\ &= \frac{1-2\cos 2x+\cos^2 2x}{4}\\ &= \frac{1-2\cos 2x+\frac{1+\cos 4x}{2}}{4}\\ &= \frac{3}{8}-\frac{\cos 2x}{2}+\frac{\cos 4x}{8}\end{aligned}$$

8.9 Half-Angle Formulas

If we put $\theta = \frac{x}{2}$ in the alternative forms of the double angle formulas (48) and (49) and take the square root, we get the half-angle formulas,

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad (50)$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad (51)$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad (52)$$

As usual, we get (52) by dividing (50) by (51).

Example 25
Find the exact value of $\sin 105^0$
Solution
Note $105^0 = 180^0 - 75^0$ is in the second quadrant where sine is positive.
Also, $105^0 = \frac{210^0}{2}$ and $210^0 = 180^0 + 30^0$ is in the third quadrant where cosine is negative.

$$\begin{aligned}\sin 105^0 &= \sin\frac{210^0}{2}\\ &= +\sqrt{\frac{1-\cos 210^0}{2}}\\ &= \sqrt{\frac{1-\cos(180^0+30^0)}{2}}\\ &= \sqrt{\frac{1+\cos 30^0}{2}}\\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Example 26
Solve $\cos 2x = 3\cos x + 1,\ 0 \le x \le 2\pi$
Solution

$$\cos 2x = 3\cos x + 1 \Leftrightarrow 2\cos^2 x - 1 - 3\cos x - 1 = 0$$

$$\Leftrightarrow (2\cos x + 1)(\cos - 2) = 0 \Leftrightarrow \cos x = -\frac{1}{2}, \not{2}$$

$$\Rightarrow x = \pi \pm \frac{\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

8.10 Product to Sum Formulas

As an exercise, you are asked to add and subtract appropriate pairs of addition formulas to get,

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \quad (53)$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \quad (54)$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \quad (55)$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \quad (56)$$

Example 27

Write $\sin\frac{\pi}{3}\cos\frac{\pi}{6}$ as a sum

Solution

$$\sin\frac{\pi}{3}\cos\frac{\pi}{6} = \frac{1}{2}\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{6}\right)\right]$$

8.11 Sum to Product Formulas

As a final exercise, by making the substitutions $A = \frac{x+y}{2}, B = \frac{x-y}{2}$ in (53) to (56) above derive the sum to product formulas,

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \quad (57)$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \quad (58)$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \quad (59)$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \quad (60)$$

Example 28
Write $\sin 10x + \sin 6x$ as a product.
Solution

$$\sin 10x + \sin 6x = 2\sin\frac{10x+6x}{2}\cos\frac{10x-6x}{2} = 2\sin 8x\cos 2x$$

Example 29
Find the exact value of $\cos 195^0 + \cos 105^0$
Solution

$$\begin{aligned}
&\cos 195^0 + \cos 105^0 \\
&= 2\cos\frac{195^0+105^0}{2}\cos\frac{195^0-105^0}{2} \\
&= 2\cos 150^0\cos 45^0 \\
&= -2\cos 30^0\cos 45^0 \\
&= -2\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{2}
\end{aligned}$$

Example 30
Verify the identity $\dfrac{\sin 3x - \sin x}{\cos x + \cos 3x} = \tan x$
Solution

$$\begin{aligned}
LS &= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)} \\
&= \frac{2\cos 2x\sin x}{2\cos 2x\cos(-x)} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x \\
&= RS
\end{aligned}$$

Exercises 8F

Solve each equation in the interval $\left[0, 2\pi\right]$

1. $\sin 2x = \sqrt{3}/2$
2. $\cos 2x = \sqrt{2}/2$
3. $\cos 4x = -\sqrt{3}/2$
4. $\sin 4x = -\sqrt{3}/2$
5. $\tan 3x = \sqrt{3}/3$
6. $\tan 3x = \sqrt{3}$
7. $\tan \frac{x}{2} = \sqrt{3}$
8. $\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$
9. $\sin \frac{2x}{3} = -1$
10. $\cos \frac{2x}{3} = -1$
11. $\sec \frac{3x}{2} = -2$
12. $\cot \frac{3x}{2} = -\sqrt{3}$
13. $\sin\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$
14. $\sin\left(2x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
15. $\sqrt{3}\tan 3x + 1 = 0$
16. $\sec 4x - 2 = 0$
17. $\cos \frac{x}{2} - 1 = 0$
18. $\tan \frac{x}{4} + \sqrt{3} = 0$
19. $2\sin \frac{x}{3} + \sqrt{3} = 0$
20. $\sec \frac{x}{2} = \cos \frac{x}{2}$
21. $2\cos^2 x + \sin x - 1 = 0$
22. $\sin 2x = \cos x$
23. $\sin 2x = \sin x$
24. $\cos 2x = \cos x$
25. $\cos 2x = \sin x$
26. $\cos 2x + 5\cos x + 3 = 0$

27. $\cos 2x + \cos x + 1 = 0$

28. $\sin x \cos x = \frac{\sqrt{2}}{4}$

Use an Addition or Subtraction formula to simplify the equation and find all solutions in the interval $[0, 2\pi]$

29. $\cos x \cos 3x - \sin x \sin 3x = 0$

30. $\cos x \cos 2x + \sin x \sin 2x = \frac{1}{2}$

31. $\sin 2x \cos x - \cos 2x \sin x = \frac{\sqrt{3}}{2}$

32. $\sin 3x \cos x - \cos 3x \sin x = 0$

Use an Half-Angle formula to find the exact value of each expression.

33. $\sin 22.5^0$

34. $\tan 67.5^0$

35. $\cos \frac{\pi}{8}$

36. $\tan 15^0$

37. $\sin \frac{11\pi}{12}$

38. $\cos 112.5^0$

39. Given $\sin\theta = -\frac{3}{5}$ and θ is in the third quadrant, find

$(a) \sin\frac{\theta}{2}, (b) \cos\frac{\theta}{2} \, (c) \tan\frac{\theta}{2}$

40. Given $\cos\theta = \frac{3}{4}$ and θ is in quadrant 4, find $(a) \sin\frac{\theta}{2}, (b) \cos\frac{\theta}{2} \, (c) \tan\frac{\theta}{2}$

Use a Double- or Half-angle formula to find all solutions in the interval $[0, 2\pi]$

41. $\sin 2x + \cos x = 0$

42. $\cos 2x + \cos x = 2$

43. $\cos 2x - \cos^2 x = 0$

44. $\cos 2\theta - \cos 4\theta = 0$

45. $\cos x - \sin x = \sqrt{2} \sin\frac{x}{2}$

Solve the equation by first using a sum-to-product formula.

46. $\sin x + \sin 3x = 0$
47. $\cos 5x - \cos 7x = 0$
48. $\cos 4x + \cos 2x = \cos x$
49. $\sin 5x - \sin 3x = \cos 4x$

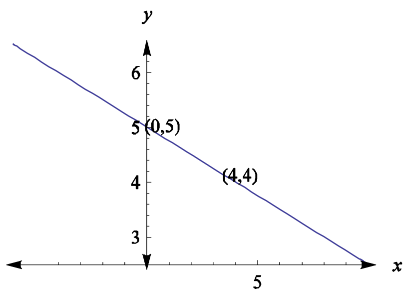
Prove the following identities

50. $\sin^2 \frac{x}{2} = \frac{\sec x - 1}{2\sec x}$
51. Show $\cos 100^0 - \cos 200^0 = \sin 50^0$
52. Show $\cos 87^0 + \cos 33^0 = \sin 63^0$

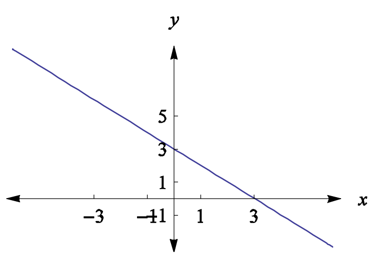
Answers to Exercises

Answers Exercises 1A

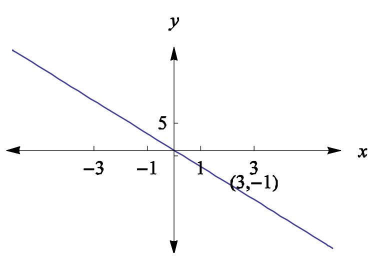
1. 15
2. $3\sqrt{2}$
3. $(-2,5), 6$
4. $(3,4), 4$
5. $x^2 + y^2 + 6x + 8y = 0$
6. $(x-3)^2 + (y-3)^2 = 85$
7. $m = 4/3$
8. 6
9.



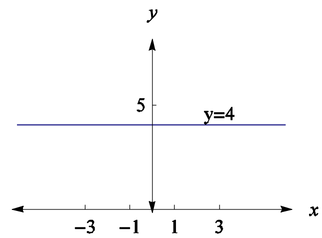
10. $3x - y + 10 = 0$
11. $x - y = 1$
12. $9x + 5y = 0$
13. $2x - 9y + 27 = 0$
14. $5x + 8y + 15 = 0$
15. $y = -3x + 7, m = -3, b = 7$
16. $m = -1, b = 3$



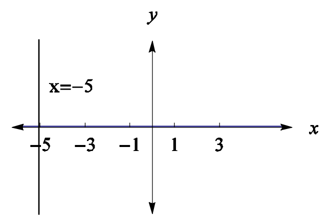
17. $m = -1/3, b = 0$



18. $m = 0, b = 4$



19. m undefined, no $y-\text{int}\,ercept$

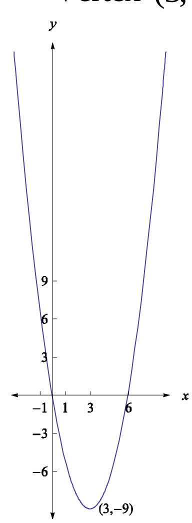


20. $(a)\ x + y + 1 = 0, (b)\ x - y + 5 = 0$
21. $(a)\ 40x + 24y = 53\ (b)\ 24x - 40y + 9 = 0$

Answers Exercises 1B

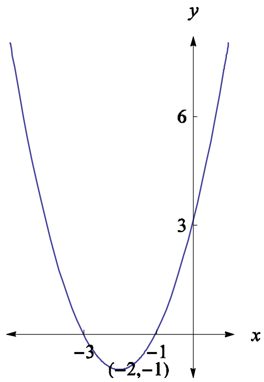
1. Intercepts $(0,0), (6,0)$

Vertex $(3,-9)$



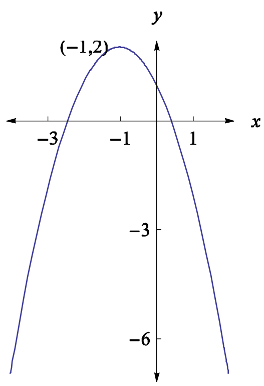
2. Intercepts $(-3,0), (-1,0), (0,3)$

Vertex $(-2,-1)$



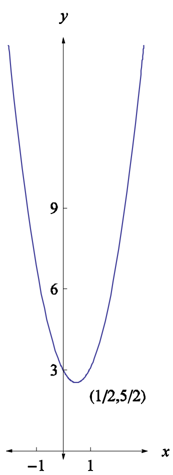
3. Intercepts $(1\pm\sqrt{2},0),(0,3)$

Vertex $(-1,2)$



4. Intercepts $(0,3)$

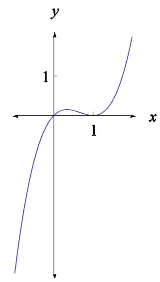
Vertex $(1/2,5/2)$



5. $(-\infty,-2)\cup(1,\infty)$

6. $(-\infty,-\frac{7}{3})\cup(\frac{1}{2},\infty)$
7. $(-\infty,-2]\cup[0,4]$
8. $(-7,5)$
9. $(-\infty,-2)\cup(2,\infty)$
10. $(-\infty,-1)\cup(2,\infty)$
11. $(-\infty,1)\cup[2,\infty)$
12. $(-6,-3)$
13. $(-\infty,5)\cup[9,\infty)$

14.



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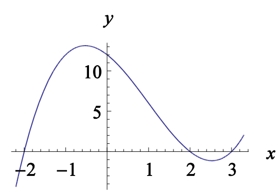

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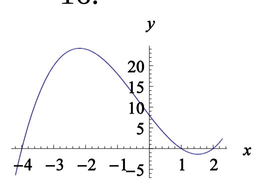
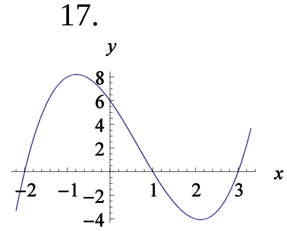


$[-4,1]\cup[2,\infty)$

17.



$(-\infty,-2]\cup[1,3]$

Answers Exercises 2A

1. (a) Y (b) N (c) Y (d) N

2. (a) Y (b) Y (c) Y (d) N

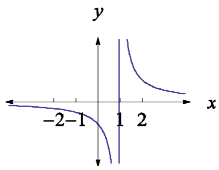
3. Yes

4. No, $y=1\pm\sqrt{1-x^2}$

5. Hor. $y=1$, Vert. $x=2$

6. Hor. $y=0$, Vert. $x=2$

7.



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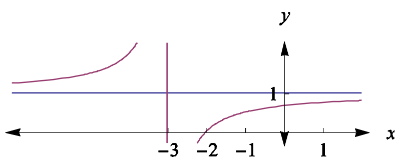

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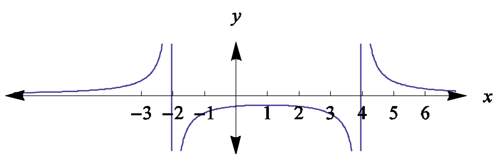
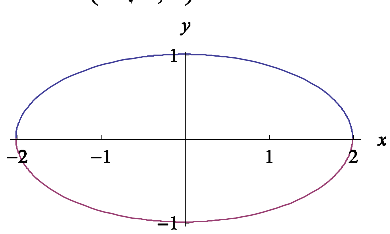


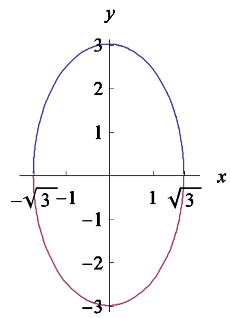
10.
$Verts:(\pm 2,0),\ Minor\ ends(0,\pm 1),$

$Foci:(\pm\sqrt{3},0)$



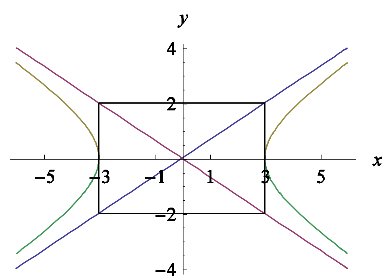
11.
$Verts:(0,\pm 3),\ Minor\ ends(\pm\sqrt{3},0),$

$Foci:(0,\pm 2\sqrt{3})$



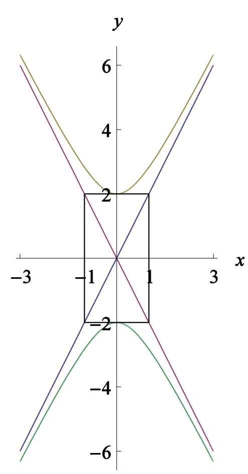
12.
$Verts:(\pm 3,0),\ Foci(\pm\sqrt{13},0),$

$Asymps: y=\pm 2x/3$



13.
$Verts:(\pm 1,0),\ Foci(\pm\sqrt{5},0),$

$Asymps: y=\pm 2x$



Answers Exercises 2B

1. $6x-2, \mathbb{R}, 6x-1, \mathbb{R}$

2. $3x^2+9x-16, \mathbb{R}, 9x^2-15x, \mathbb{R}$

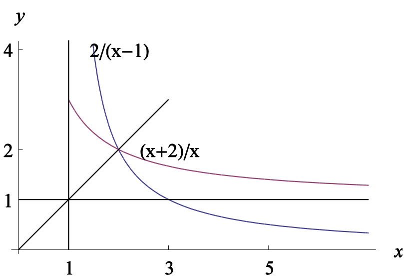
3. $1/(2x+7), \{x : x \neq -7/2\}$
4. $\sqrt{3x-3}, \{x : x \geq 1\}$
   $3\sqrt{x-2}-1, \{x : x \geq 2\}$
5. $x/(2-x), \{x : x \neq 0, 2\}$
   $2x-2, \{x : x \neq 1\}$
6. $2\sqrt{x-1}+1, \{x : x \geq 1\}$
   $\sqrt{2x}, \{x : x \geq 0\}$
7. 34,18

8. $f^{-1}(x) = x+4$

9. $f^{-1}(x) = (12x+10)/9$

10. $f^{-1}(x) = x^2, x \geq 0$

11. $f^{-1}(x) = (x+2)/x = 1+2/x$



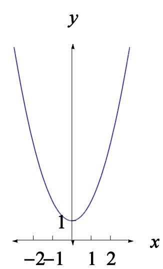
12. Domain of f : $\{x : x \geq 5\}$
    Range of f : $\{x : x \geq 0\}$
    $f^{-1}(x) = x+5$
    Domain and range of f^{-1} is $\mathbb{R}$

13. Domain of f : $\{x : x \geq 4\}$
    Range of f : $\{x : x \geq 0\}$
    $f^{-1}(x) = \sqrt{x}+4$
    Range of f^{-1} : $\{x : x \geq 4\}$
    Domain of f^{-1} : $\{x : x \geq 0\}$

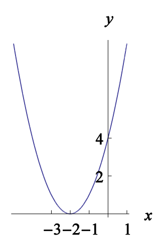
14. Domain of f : $\{x : x \geq 0\}$
    Range of f : $\{x : x \geq -1\}$
    $f^{-1}(x) = \sqrt{2x+2}$
    Domain of f^{-1} : $\{x : x \geq -1\}$
    Range of f^{-1} : $\{x : x \geq 0\}$

Answers Exercises 3A

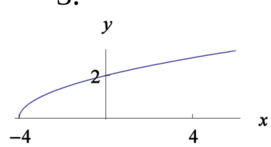
1.



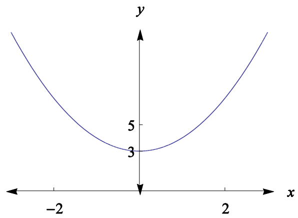
2.



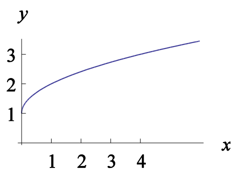
3.



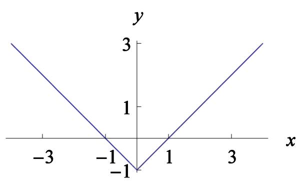
4.



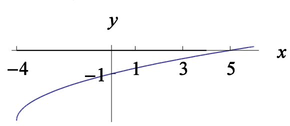
5.



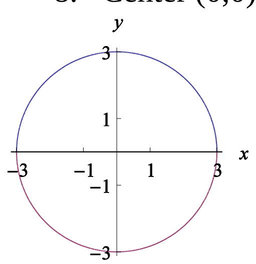
6.



7.

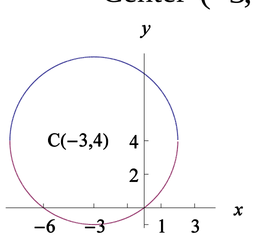


8. Center (0,0), radius 3

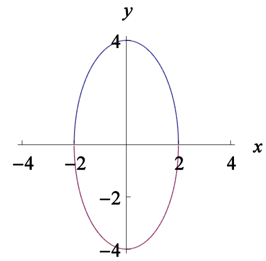


9.

Center $(-3,4)$, radius 5

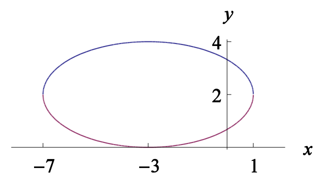


10.



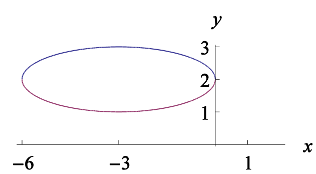
Vertices $(0,\pm 4)$
Major $(0,\pm 4)$
Minor $(\pm 2,0)$
Foci $(0,\pm 2\sqrt{3})$

11.



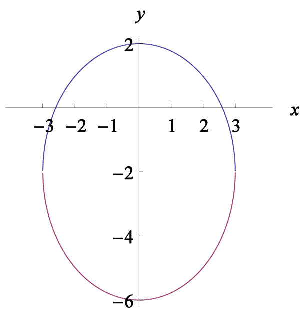
Vertices $(1,2),(-7,2)$
Major $(1,2),(-7,2)$
Minor $(-3,4),(-3,0)$
Foci $(4\pm 2\sqrt{3},2)$

12.



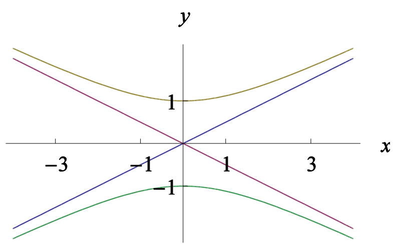
Vertices $(0,2),(-6,2)$
Major $(0,2),(-6,2)$
Minor $(-3,3),(-3,1)$
Foci $(-3\pm 2\sqrt{2},2)$

13.



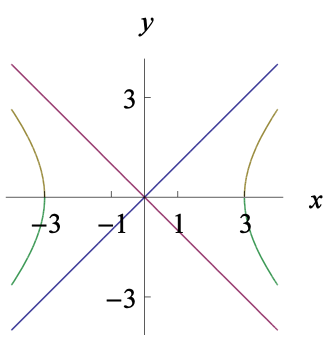
Vertices $(0,2),(0,-6)$
Major $(0,2),(0,-6)$
Minor $(-3,-2),(3,-2)$
Foci $(0,2\pm\sqrt{7})$

14. Center $(-3,4)$, radius 5



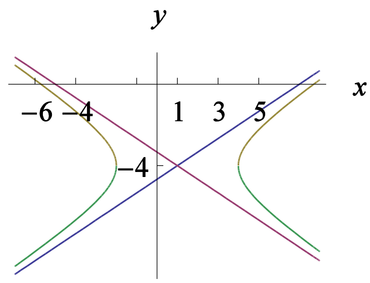
Vertices $(0,\pm 1)$
Foci $(0,\pm\sqrt{5})$
Asymp: $y=\pm x/2$

15.



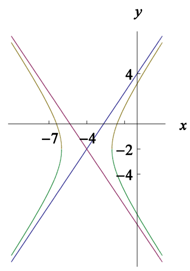
Vertices $(\pm 3,0)$
Foci $(\pm\sqrt{2},0)$
Asymp: $y=\pm x$

16.



Vertices $(-2,-4),(4,-4)$
Foci $(1\pm\sqrt{13},-4)$
Asym: $y+4=\pm\dfrac{2}{3}(x-1)$

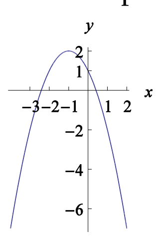
17.



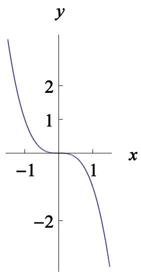
Foci $(-4\pm\sqrt{13},-2)$
Vertices $(-6,-2),(-2,-2)$
Asym: $y+2=\pm\dfrac{3}{2}(x+4)$

Answers Exercises 3B

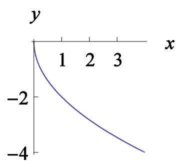
1. vertex $(-1,2)$
y- intercept (0,1)
x-intercepts $(1\pm\sqrt{2},0)$



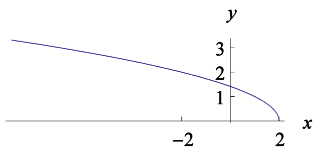
2.



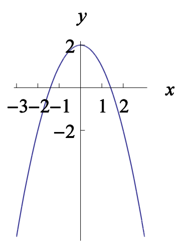
3.



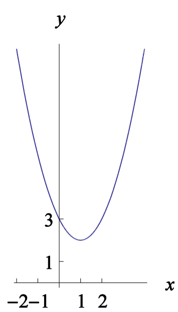
4.



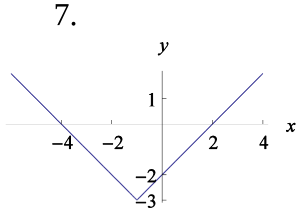
5.



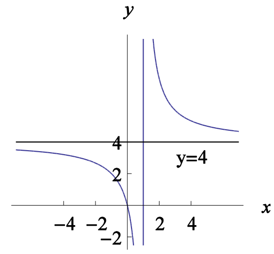
6.



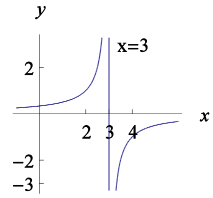
7.



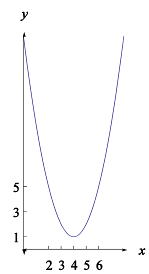
8.



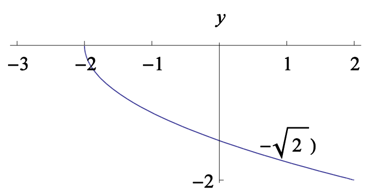
9.



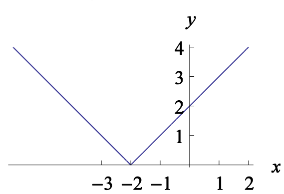
10.



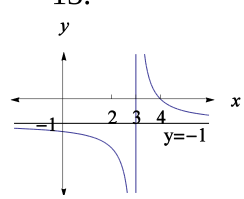
11.



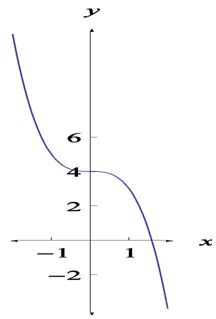
12.



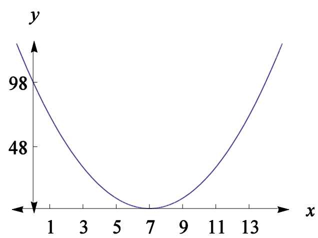
13.



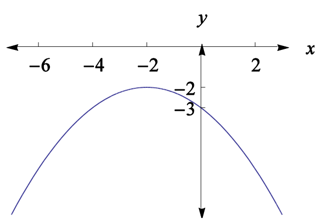
14. .



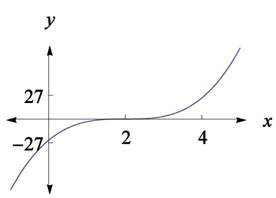
15.



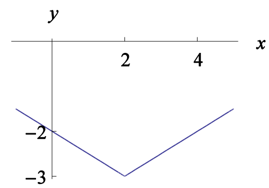
16.



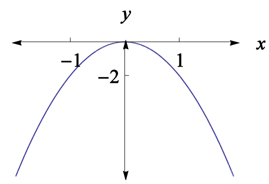
17.



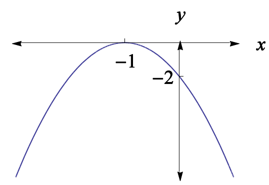
18.



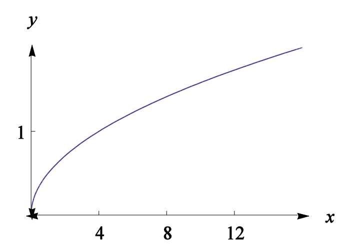
19.



20.

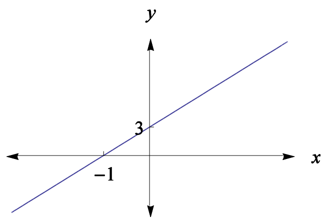


21.



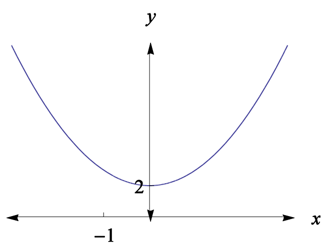
Answers Exercises 3C

1.



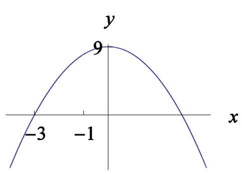
No symmetries

2.



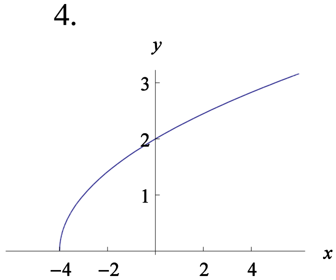
y-axis symmetry

3.



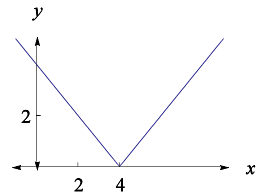
y-axis symmetry

4.



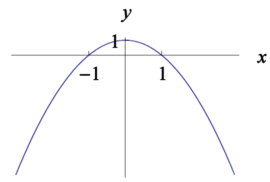
No symmetries

5.

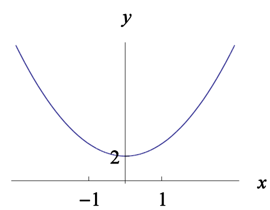


No symmetries

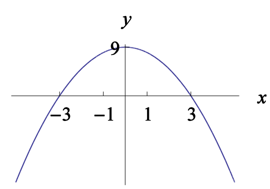
6.



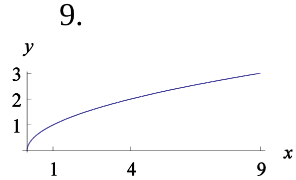
7.



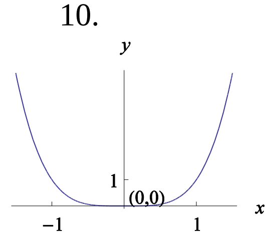
8.



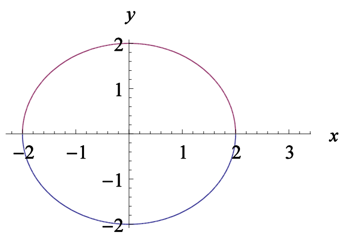
9.



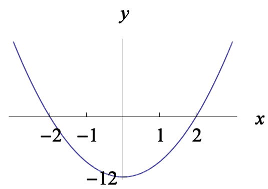
10.



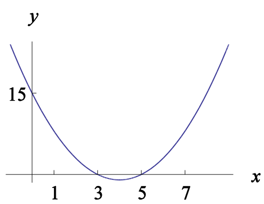
11.



12.



13.



14. x-axis symmetry

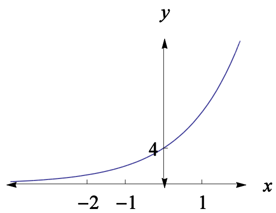
15. no symmetries

16. (a) C, (b) F, (c) I, (d) A, (e) J, (f) D, (g) B, (h) E, (i) H, (j) G, (k) K

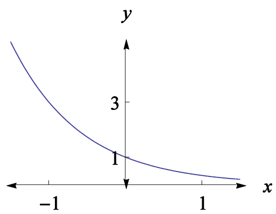
17. (a) K, (b) B, (c) J, (d) L, (e) A, (f) C, (g) E, (h) D, (i) H, (j) G, (k) I, (l) F

Answers Exercises 4A

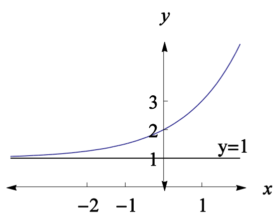
1.



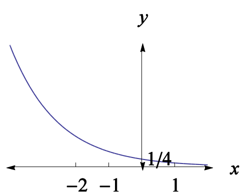
2.



3.



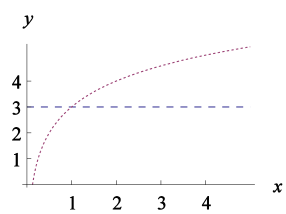
4.



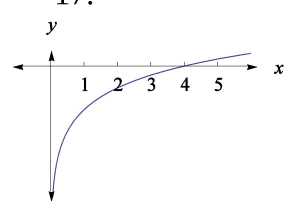
6. {3/2}
7. {7}
8. {1}
9. {1/5}
10. {-1}
11. 6
12. {5}
13. {3/2}
14. {0}
15. {3}
16. {5/2}
17. $283.70
18. $1251.59
19. $567.63
20. $760.98
21. 625 mg, 442 mg
22. 354 grams
23. 2226, 3320, 7389

Answers Exercises 4B

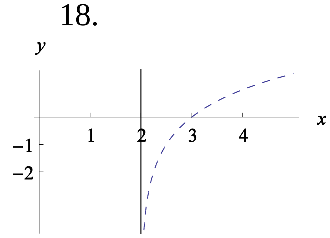
1. $4^3 = 64$
2. $10^4 = 10,000$
3. $10^{-3} = 0.001$
4. $\log_2 16 = 4$
5. $\log_{\frac{1}{3}} \frac{1}{729} = 6$
6. 4
7. ½
8. 5
9. 1
10. 4
11. -3
12. 3
13. 16
14. 27
15. 3
16.



17.



18.



19. 1.6094
20. -3.114

Answers Exercises 4C

1. -3
2. 1/3
3. -12
4. 7
5. $\log_5 \frac{8}{t}$
6. $\log_2 x^2 y^4$
7. $\log_6 \frac{1}{16x^4}$
8. $\ln 64(z-4)^5$
9. $\ln\left(\frac{x^3}{x^2-1}\right)^2$
10. $\log_3 10 + \log_3 z$
11. $\log y - \log 2$
12. $\ln(x+1)+\ln(x-1)-3\ln x$
13. $\ln x - \frac{3}{2}\ln y$
14. $4 + 2\log_2 3$
15. $\log 3 - 2$
16. 0.7183
17. 1004
18. -25
19. -2, 3
20. 2, 4
21. 4
22. 4
23. 19/47
24. $\left\{\frac{-1+\sqrt{33}}{4}\right\}$
25. 2.33
26. 5.43
27. 4.57
28. 3.32
29. 1.70
30. -29.34
31. 0.05
32. -3.10
33. 0.712
34. -1.161
35. -0.463

Answers Exercises 5A

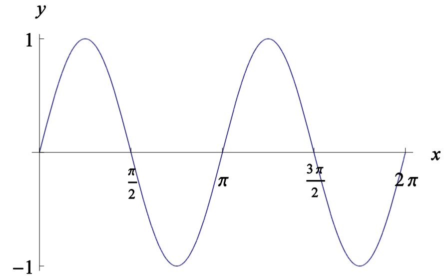
1. (a) 2nd (b) 4th
2. .
3. (a) $7\pi/4$ (b) $2\pi/3$
4. (a) -150^0 (b) 15^0
5. (a) -330^0 (b) 168^0
6. (a) 1.210 (b) -1.764
7. 3π
8. 2.5
9. 0.8
10. 274.9 sq.m.
11. $\sin\theta = -5/13, \cos\theta = 12/13, \tan\theta = -5/12$
$\csc\theta = -13/15, \sec\theta = 13/12, \cot\theta = -12/5$
12. $(-1, 0)$
13. $(-\sqrt{2}/2, \sqrt{2}/2)$
14. $\sin\pi/3 = \sqrt{3}/2,\ \cos\pi/3 = 1/2,$
$\tan\pi/3 = \sqrt{3}$
15. $-\sqrt{2}/2, \sqrt{2}/2, -1$
16. $\sqrt{3}/2,\ -1/2,\ -\sqrt{3}$
17. $-\sqrt{3}/2, 1/2, -\sqrt{3}$
18. $\sin\theta = 1/2,\ \cos\theta = -\sqrt{3}/2,$
$\csc\theta = 2, \sec\theta = -2\sqrt{3}/3,$
$\tan\theta = -\sqrt{3}/3,\ \cot\theta = -\sqrt{3}$
19. $\sin\theta = -\sqrt{2}/2,\ \cos\theta = \sqrt{2}/2,$
$\csc\theta = -\sqrt{2}, \sec\theta = \sqrt{2}$
$\tan\theta = -1,\ \cot\theta = -1$
20. $\sin\theta = -1,\ \cos\theta = 0, \tan\theta = undefined$
$\csc\theta = -1, \sec\theta = undefined, \cot\theta = 0$
$\sin\theta = 0,\ \cos\theta = -1, \tan\theta = 0$
21. $\csc\theta = undefined, \sec\theta = -1,$
$\cot\theta = undefined$
22. -1
24. $\sqrt{2}/2$
25. (a) $-3/8$ (b) $-8/3$
26. $-1/2$

Answers Exercises 5B

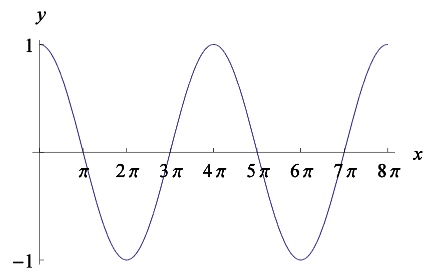
1. $\pi/3+2n\pi, 2\pi/3+2n\pi,\ n\in\mathbb{Z}$
2. $2\pi/3+n\pi, 5\pi/3+n\pi,\ n\in\mathbb{Z}$
3. $5\pi/6+2n\pi, 7\pi/6+2n\pi, n\in\mathbb{Z}$
4. $5\pi/4+2n\pi, 7\pi/4+2n\pi,\ n\in\mathbb{Z}$
5. $\pi/4+2n\pi, 7\pi/4+2n\pi,\ n\in\mathbb{Z}$
6. $\pi/4+n\pi, 3\pi/4+n\pi, 5\pi/4+n\pi, 7\pi/4+n\pi,\ n\in\mathbb{Z}$
7. $\pi/3+2n\pi, 5\pi/3+2n\pi,\ n\in\mathbb{Z}$
8. $7\pi/6+2n\pi, 11\pi/6+2n\pi, \pi/2+2n\pi,\ n\in\mathbb{Z}$
9. $7\pi/6+2n\pi, 11\pi/6+2n\pi,\ n\in\mathbb{Z}$
10. $2\pi/3+n\pi, 5\pi/3+n\pi,\ n\in\mathbb{Z}$
11. $\pi/3+n\pi, 2\pi/3+n\pi, 4\pi/3+n\pi, 5\pi/3+n\pi, \pi/6+n\pi, 5\pi/6+n\pi, 7\pi/6+n\pi, 11\pi/6+n\pi,\ n\in\mathbb{Z}$
12. $\pi/18+n\pi/3, 5\pi/18+n\pi/3, 7\pi/18+n\pi/3, 11\pi/18+n\pi/3,\ n\in\mathbb{Z}$
13. $2n\pi, \pi+2n\pi,\ n\in\mathbb{Z}$ or simply $n\pi, n\in\mathbb{Z}$
14. $7\pi/6+2n\pi, 11\pi/6+2n\pi, 3\pi/2+2n\pi,\ n\in\mathbb{Z}$

Answers Exercises 6A

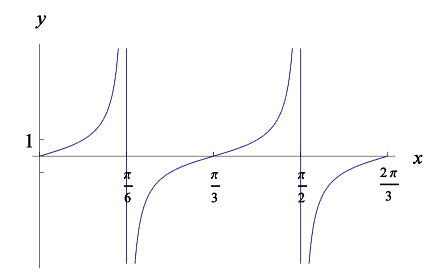
1. Amplitude: 1, Period: π, Shift: none



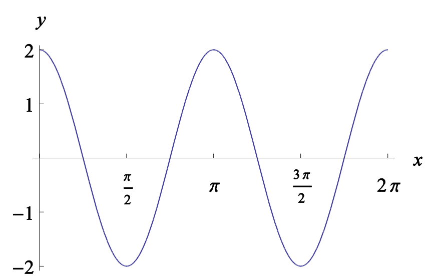
2. Amplitude:1, Period: 4π, Shift: None



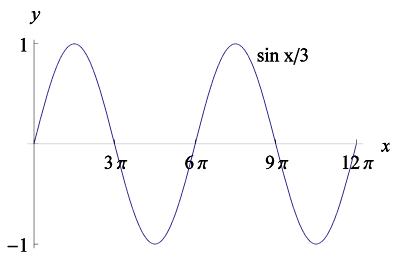
3. Amplitude: ∞, Period: $2\pi/3$, Shift: None



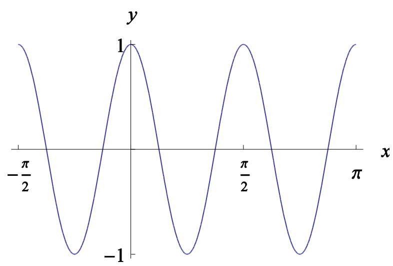
4. Amplitude: 2, Period: π, Shift: None



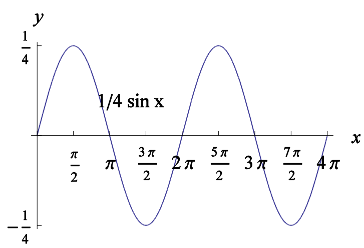
5. Amplitude: 1, Period: 6π, Shift: None



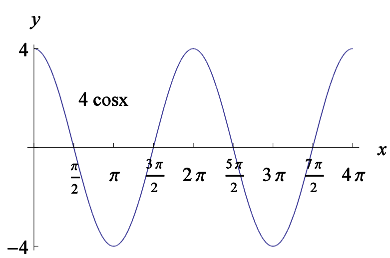
6. Amplitude: 1, Period: $\pi/2$, Reflected in y-axis (causes no change)



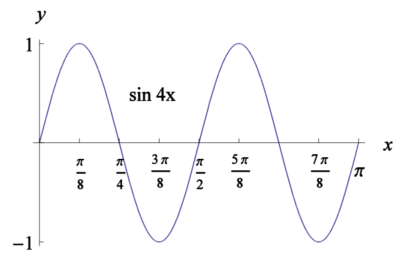
7. Amplitude: 1/4, Period: 2π, Shift: None



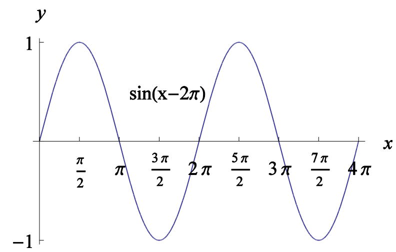
8. Amplitude: 4, Period: 2π, Shift: None



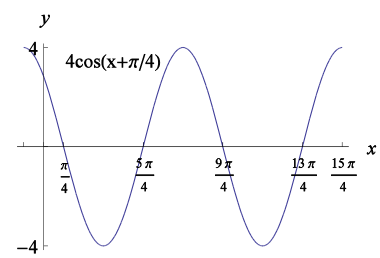
9. Amplitude: 1, Period: $\pi/2$, Shift: None



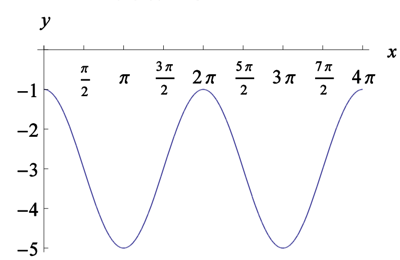
10. Amplitude: 1, Period: 2π, Phase shift: 2π right (no resultant change)



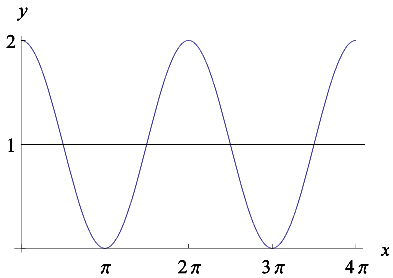
11. Amplitude: 4, Period: 2π, Phase shift: $\pi/4$ left



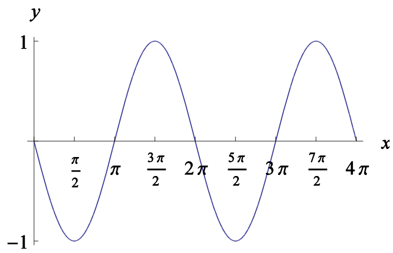
12. Amplitude: 2, Period: 2π, Phase shift: None, Vertical shift: down 3



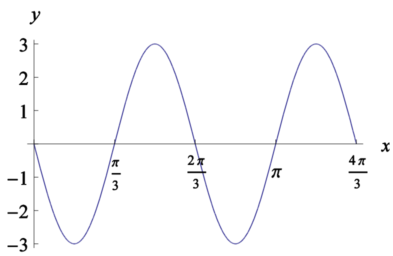
13. Amplitude: 1, Period: 2π, Phase shift: None, Vertical shift: up 1



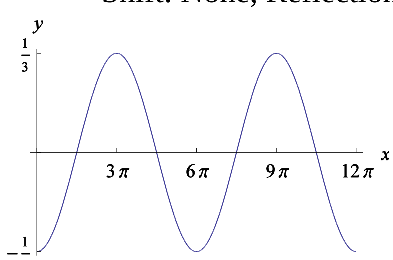
14. Amplitude: 1, Period: 2π , Phase shift: None, Reflection in x-axis



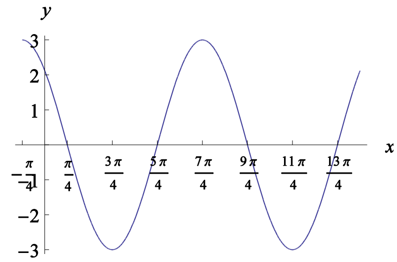
15. Amplitude $=3$, Period $=2\pi/3$, Phase shift: None, Reflection in x-axis



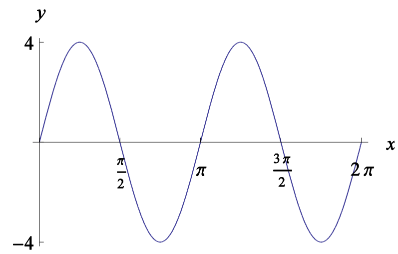
16. Amplitude $=1/3$, Period $=6\pi$, Shift: None, Reflection in x-axis



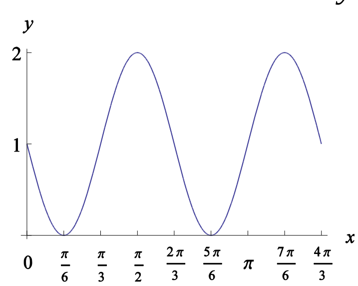
17. Amplitude $=3$, Period $=2\pi$, Shift $(0,0)\rightarrow(-\pi/4,0)$ or shifts $\pi/4$ to left



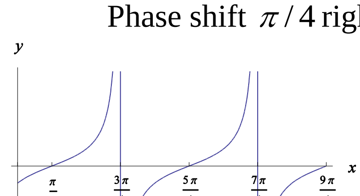
18. Amplitude $=4$, Period $=\pi$, Shift $(0,0)\rightarrow(-\pi/2,0)$ or shifts $\pi/2$ to left . Reflection in x-axis. Ends up same as $y=\sin x$



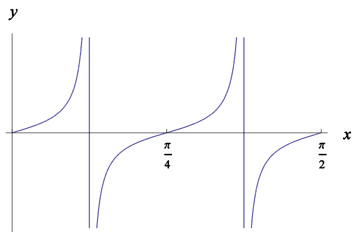
19. Amplitude $=1$, Period $=2\pi/3$, Shift $(0,0)\rightarrow(-\pi/6,1)$ or shifts $\pi/6$ to left and up 1 so it oscillates around $y=1$.



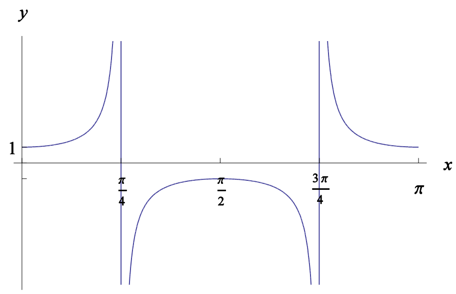
20. Amplitude: ∞, Period = π , Phase shift $\pi/4$ right



21. Amplitude: ∞, Period = $\pi/2$, Phase shift: None

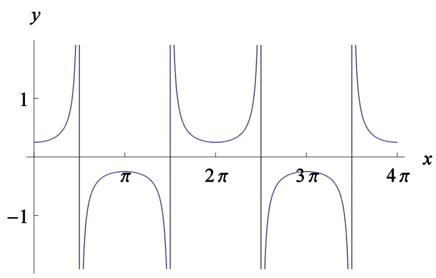


22. Period = π



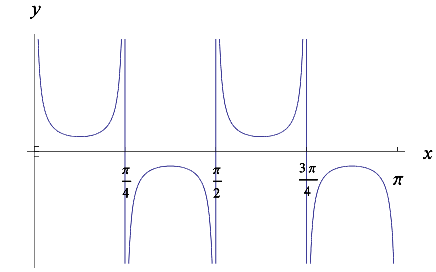
Answers Exercises 6B

1.

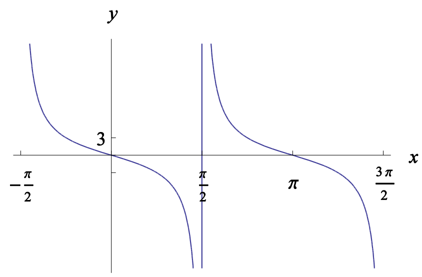


4. $\pi/6$
5. $\pi/4$
6. $-\pi/6$
7. $\pi/3$
8. $3\pi/4$
9. $2\pi/3$
10. π
11. $\pi/6$
12. 0
13. $-\pi/3$
14. $\pi/3$
15. $\pi/6$

2.



3.



16. 125
17. $-\pi/6$
18. $-\pi/3$
19. 0
20. $3/5$
21. $12/5$
22. $-3/4$
23. $\pi/4$
24. $-1/4$
25. $-2\sqrt{2}$
26. $\sqrt{6}/2$

Answers Exercises 7A

1. $\sin A = 5/13, \cos A = 12/13,$ $\tan A = 5/12$ $\csc A = 13/5, \sec A = 13/12,$ $\cot A = 12/5$

2. $\sin\theta = \sqrt{11}/6, \cos\theta = 5/6,$ $\tan\theta = \sqrt{11}/5$ $\csc\theta = 6\sqrt{11}/11, \sec\theta = 6/5,$ $\cot\theta = 5\sqrt{11}/11$

3. $\sin\theta = 4\sqrt{41}/41, \cos\theta = 5\sqrt{41}/41,$ $\tan\theta = 4/5$ $\csc\theta = \sqrt{41}/4, \sec\theta = \sqrt{41}/5$ $\cot\theta = 5/4$

4. $\sin\theta = 4\sqrt{15}/17, \cos\theta = 7/17,$ $\tan\theta = 4\sqrt{15}/7$ $\csc\theta = 17\sqrt{15}/60, \sec\theta = 17/7,$ $\cot\theta = 7\sqrt{15}/60$

5. $\sin\theta = 1/9, \cos\theta = 4\sqrt{5}/9,$ $\tan\theta = \sqrt{5}/4$ $\csc\theta = 9, \sec\theta = 9\sqrt{5}/20,$ $\cot\theta = 4\sqrt{5}$

6. $a = 8.82,\ b = 12.14,\ A = 36^0$
7. 29 feet
8. 36^0
9. 436 feet
10. 443.2m, 323.3m
11. 214.45 ft
12. 11.8 km
13. 474.9
14. 4.5

Answers Exercises 7B

1. $C = 105^0,\ a = 6.10,\ b = 6.84$
2. $C = 120^0,\ b = 4.55,\ c = 6.86$
3. $A = 65^0,\ a = 42.22,\ b = 8.09$
4. $B = 101.1^0,\ a = 1.35,\ b = 3.23$
5. $C = 166.5^0,\ a = 3.30,\ c = 8.05$
6. 184 feet
7. 233 feet
8. 30 feet
9. 343 feet
10. $A = 60^0,\ B = 21.79^0,\ C = 98.21^0$
11. $B = 23.79^0,\ C = 126.21^0,\ a = 18.59$
12. $A = 86.68^0,\ B = 31.82^0,\ C = 61.50^0$
13. Law of cosines
    $A = 102.44^0,\ B = 62.44^0,\ c = 12.72$
14. Law of cosines
    $A = 57.79,\ B = 89.63^0,\ C = 32.58^0$

15. Law of sines
    $A = 161^0,\ b = 102.18,\ c = 59.89$
16. 61.7 miles
17. (a) 19.3 miles (b) $S58^0E$
18. 63.7 feet

Answers Exercises 8A

1. (a) $\sqrt{3}/2$ (b) $-1/2$ (c) $-\sqrt{3}$

2. (a) $-1/2$ (b) $-1/2$ (c) $-1/2$

3. (a) $1/2$ (b) 2 (c) $-\sqrt{3}/3$

4. (a) 1 (b) 0 (c) 0

5. (a) $-\sqrt{3}/3$ (b) $\sqrt{3}/3$ (c) $-\sqrt{3}/3$

6. $\sin t = 3/5, \cos t = -4/5,$ $\tan t = -3/4, \csc t = 5/3,$ $\sec t = -5/4, \cot t = -4/3$

7. $\sin t = -3/5, \cos t = -4/5,$ $\tan t = 3/4, \csc t = -5/3,$ $\sec t = -5/4, \cot t = 4/3$

8. $\sin\theta = 2\sqrt{2}/3, \cos\theta = 1/3, \tan\theta = 2\sqrt{2},$ $\csc\theta = 3\sqrt{2}/4, \sec\theta = 3, \cot\theta = \sqrt{2}/4$

9. $\sin\theta = -\sqrt{17}/17, \cos\theta = -4\sqrt{17}/17,$ $\tan\theta = 1/4, \csc\theta = -\sqrt{17},$ $\sec\theta = -\sqrt{17}/4, \cot\theta = 4$

10. $\sin\theta = -\sqrt{3}/2, \cos\theta = 1/2, \tan\theta = -\sqrt{3},$ $\csc\theta = -2\sqrt{3}/3, \sec\theta = 2, \cot\theta = -\sqrt{3}/3$

11. $\sin\theta = -1/4, \cos\theta = -\sqrt{15}/4,$ $\tan\theta = \sqrt{15}/15, \csc\theta = -4,$ $\sec\theta = -4\sqrt{15}/15, \cot\theta = \sqrt{15}$

---

12. $\sin t$
13. $\cot t$
14. $\tan\theta$
15. -1
16. $\tan x$
17. $\cos x$
18. $\cos^2\phi$
19. $\sec x$
20. 1
21. $\cot x$
22. $\csc u$
23. 1
24. $\tan\theta$
25. $\sec\theta$
26. 1
27. $1+\sin y$
28. $\csc\theta$
29. $\sec\theta + \tan\theta$ *or* $\dfrac{1+\sin\theta}{\cos\theta}$
30. $2\csc\theta$
31. $\sec\theta\csc\theta$ *or* $\dfrac{1}{\sin\theta\cos\theta}$
32. $\cos x$
33. $\cos y$
34. $\sin x$
35. $\sin^2 x$
36. $\sin x$
37. $\cos^2\theta$
38. $-\cos\theta$
39. $2\cot\theta$
40. $1+2\sin x\cos x$
41. $2\csc^2 x$
42. $\sec x$
43. 1
44. $2\sec u$
45. 1
46. $\cos^2 x$
47. $\sin A + \cos A$
48. $\cos\theta$
49. $1-\sin x$

There are no answers required for Exercise 8B on proving identities.

Answers Exercises 8C

1. $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$
2. $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

3. $\frac{\pi}{2}$

4. $\frac{3\pi}{4}, \frac{7\pi}{4}$

5. $\frac{\pi}{2}$

6. $\frac{2\pi}{3}, \frac{5\pi}{3}$

7. no solution

8. $\pi/3, 2\pi/3$

9. π

10. $\pi/6, 11\pi/6$

11. $\pi/4, 5\pi/4$

12. $\pi/4, 3\pi/4$

13. π

14. 0.20, 2.94

15. $\pi/2, 3\pi/2$

16. $\pi/6, 5\pi/6, 7\pi/6, 11\pi/6$

17. $\pi/3, 2\pi/3, 4\pi/3, 5\pi/3$

18. $0, 2\pi$

19. $0, \pi, 2\pi, \pi/4, 5\pi/4$

20. $0, \pi, 2\pi$

21. $\pi/4, 5\pi/4$

22. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

23. $-1.11, 1.11$

24. $\pi/4, 3\pi/, 5\pi/4, 7\pi/4$

25. $-1.11, 1.11, 2\pi/3, 4\pi/3$

26. 0.34, 2.80

27. $\pi/3, 5\pi/3$

28. No solutions

29. $\pi/6, 5\pi/6, 3\pi/2$

30. $0, \pi/2$

31. 0

32. $3\pi/2 + 2n\pi \quad, n \in \mathbb{Z}$

33. $\begin{cases} \pi/2 + n\pi, 7\pi/6 + 2n\pi, \\ 11\pi/6 + 2n\pi,\ n \in \mathbb{Z} \end{cases}$

34. $\pi/2 + n\pi,\ n \in \mathbb{Z}$

35. $\begin{cases} n\pi, 0.73 + 2n\pi, 2.41 + 2n\pi, \\ n \in \mathbb{Z} \end{cases}$

36. $\begin{cases} \pi/2 + 2n\pi, 7\pi/6 + 2n\pi, \\ 11\pi/6 + 2n\pi,\ n \in \mathbb{Z} \end{cases}$

37. No solutions

38. $\begin{cases} 1.23 + 2n\pi, 5.05 + 2n\pi, \\ \pi + 2n\pi,\ n \in \mathbb{Z} \end{cases}$

39. $-\pi/4 + n\pi, 0.46 + n\pi,\ n \in \mathbb{Z}$

40. $\begin{cases} n\pi,\ 0.72 + 2n\pi, \\ 5.56 + 2n\pi,\ n \in \mathbb{Z} \end{cases}$

41. $n\pi, 1.23 + 2n\pi, 5.05 + 2n\pi, n \in \mathbb{Z}$

42. $\pi/6 + 2n\pi, 5\pi/6 + 2n\pi,\ n \in \mathbb{Z}$

43. $\begin{cases} \pi/4 + 2n\pi, 3\pi/4 + 2n\pi, 5\pi/4 + 2n\pi, \\ 7\pi/4 + 2n\pi = \pi/4 + n\pi, 3\pi/4 + n\pi, \\ n \in \mathbb{Z} \end{cases}$

44. $\pi + 2n\pi, \pi/3 + 2n\pi, 5\pi/3 + 2n\pi,\ n \in \mathbb{Z}$

45. $\pi/3 + n\pi, 2\pi/3 + n\pi,\ n \in \mathbb{Z}$

46. $n\pi,\ \pi/2 + 2n\pi,\ n \in \mathbb{Z}$

47. $n\pi,\ 3\pi/2 + 2n\pi,\ n \in \mathbb{Z}$

48. $n\pi,\ n \in \mathbb{Z}$

49. $-1.24 + n\pi,\ \pi/4 + n\pi,\ n \in \mathbb{Z}$

50. $\pi/9 + 2n\pi/3,\ 5\pi/9 + 2n\pi/3,\ n \in \mathbb{Z}$

51. $\begin{cases} \pi/3 + 2n\pi, 2\pi/3 + 2n\pi, 4\pi/3 + 2n\pi, \\ 5\pi/3 + 2n\pi,\ n \in \mathbb{Z} \end{cases}$

52. $\pi/3 + n\pi, 2\pi/3 + n\pi,\ n \in \mathbb{Z}$

53. $7\pi/18 + 2n\pi/3, 11\pi/18 + 2n\pi/3,\ n \in \mathbb{Z}$

Answers Exercises 8D

1. $\frac{\sqrt{6}+\sqrt{2}}{4}$

2. $\frac{\sqrt{6}-\sqrt{2}}{4}$

3. $-\frac{\sqrt{6}-\sqrt{2}}{4}$

4. $-\frac{\sqrt{6}+\sqrt{2}}{4}$

5. $1-\sqrt{3}$
6. $-2+\sqrt{3}$
7. $-\frac{\sqrt{6}+\sqrt{2}}{4}$
8. $-\frac{\sqrt{6}-\sqrt{2}}{4}$
9. $\sqrt{3}-1$
10. $-\frac{\sqrt{6}+\sqrt{2}}{4}$
11. $-\frac{\sqrt{6}+\sqrt{2}}{4}$
12. $-(2+\sqrt{3})$
13. $\frac{1}{2}$
14. $-\frac{\sqrt{3}}{2}$
15. $1/2$
16. $\frac{\sqrt{3}}{3}$
17. $\frac{\sqrt{3}}{3}$
18. $-1/2$
19. 0
20. 1
21. $\frac{\sqrt{3}}{2}$
22. -1
37. $-\frac{3+4\sqrt{3}}{10}$
33. $-3\sqrt{10}/10$
34. $2\sqrt{5}/65$
35. $-\frac{32\sqrt{2}+9\sqrt{15}}{7}$

Answers Exercises 8E

1. $119/169$
2. $15/8$
3. $-336/625$
4. $-3/4$
5. $-24/25,\ -7/25,\ 24/7$
6. $-\sqrt{15}/8, 7/8, -\sqrt{15}/7$
7. $-\sqrt{3}/2,\ -1/2,\ \sqrt{3}$
8. $-12/13, -5/13, 12/5$
9. $\frac{\cos 4x}{8}+\frac{\cos 2x}{2}+\frac{3}{8}$

Answers Exercises 8F

1. $\pi/6, \pi/3, 7\pi/6, 4\pi/3$
2. $\pi/8, 7\pi/8, 9\pi/8, 15\pi/8$
3. $5\pi/24, 7\pi/24, 17\pi/24, 19\pi/24, 29\pi/24, 31\pi/24, 41\pi/24, 43\pi/24$
4. $\pi/3, 5\pi/12, 5\pi/6, 11\pi/12, 4\pi/3, 17\pi/12, 11\pi/6, 23\pi/12$
5. $\pi/18, 7\pi/18, 13\pi/18, 19\pi/18, 25\pi/18, 31\pi/18$
6. $\pi/9, 4\pi/9, 7\pi/9, 10\pi/9, 13\pi/9, 16\pi/9$
7. $2\pi/3$
8. $\pi/3$
9. No solution in $[0, 2\pi]$
10. $3\pi/2$
11. $4\pi/9, 8\pi/9, 16\pi/9$
12. $5\pi/9, 11\pi/9, 17\pi/9,$
13. $0, \pi/3, \pi, 4\pi/3$
14. $\pi/4, \pi/2, 5\pi/4, 3\pi/2$
15. $2\pi/9, 5\pi/9, 8\pi/9, 11\pi/9, 14\pi/9, 17\pi/9$
16. $\pi/12, 7\pi/12, 13\pi/12, 19\pi/12, 5\pi/12, 11\pi/12, 17\pi/12$
17. 0
18. No solutions in $[0, 2\pi]$
19. No solutions in $[0, 2\pi]$
20. $0, 2\pi$
21. $\pi/2, 7\pi/6, 11\pi/6$
22. $\pi/6, \pi/2, 5\pi/6, 3\pi/2$
23. $0, \pi/3, \pi, 5\pi/3$
24. $0, 2\pi/3, 4\pi/3, 2\pi$
25. $\pi/6, 5\pi/6, 3\pi/2$
26. $2\pi/3, 4\pi/3$
27. $\pi/2, 2\pi/3, 4\pi/3, 3\pi/2$
28. $\pi/8, 3\pi/8, 9\pi/8, 11\pi/8$
29. $\pi/8, 3\pi/8, 5\pi/8, 7\pi/8, 9\pi/8, 11\pi/8, 13\pi/8, 15\pi/8$
30. $\pi/3, 5\pi/3$
31. $\pi/3, 2\pi/3$
32. $0, \pi/2, \pi, 3\pi/2, 2\pi$
33. $\dfrac{\sqrt{2-\sqrt{2}}}{2}$
34. $1+\sqrt{2}$
35. $\dfrac{\sqrt{2-\sqrt{2}}}{2}$
36. $2-\sqrt{3}$

37. $\frac{\sqrt{2-\sqrt{3}}}{2}$

38. $-\frac{\sqrt{2-\sqrt{2}}}{2}$

39. $(a)\ 3/\sqrt{10}\ (b)\ -1/\sqrt{10}\ (c)\ -3$

40. $(a)\ \sqrt{2}/4\ (b)\ -\sqrt{14}/4\ (c)\ -\sqrt{7}/7$

41. $\pi/2, 3\pi/2, 7\pi/6, 11\pi/6$

42. $0, 2\pi$

43. $0, \pi, 2\pi$

44. $\pi/3, 2\pi/3, 4\pi/3, 5\pi/3, 0, \pi, 2\pi$

45. $3\pi/2, \pi/6, 5\pi/6$

46. $0, \pi/2, \pi, 3\pi/2, 2\pi$

47. $0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \pi, 7\pi/6, 4\pi/3, 3\pi/2, 5\pi/3, 11\pi/6, 2\pi$

48. $\pi/2, 3\pi/2, \pi/9, 5\pi/9, 7\pi/9, 11\pi/9, 13\pi/9, 17\pi/9$

49. $\pi/6, 5\pi/6, \pi/8, 3\pi/8, 5\pi/8, 7\pi/8, 9\pi/8, 11\pi/8, 13\pi/8, 15\pi/8$