Second Edition

Concept Builders

for

Mathematical Methods

Exploring Transformations Part 1

Jana Kohout

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Advice to teachers

Australian Mathematics educators in the early 21st century are facing some difficult questions. Why are so many of our upper secondary students abandoning the harder Maths? Why are they not connecting with this study? Has it become less relevant? Do we need more "real life" problems and more student directed activities, or are we failing on another level? I believe that Maths teaching has become more process driven at the expense of understanding, and that is where the problem lies. The quantity of material and limited time available for course delivery have made matters worse.

An appreciation of Mathematics can only come in the first instance from understanding. A failure to give sufficient attention to the teaching of concepts and logical thinking, with a focus instead on seemingly disconnected processes and rules, drains this study of the natural beauty it holds and fails to capture the imagination of students. With that in mind, this workbook was produced to provide students from year 10 upwards with the opportunity to begin exploring transformations of functions for themselves. A handy summary of the content is included in the appendix, but is not a substitute for the opportunity to explore that is provided through the exercises. Finally, as with any skill we want to master, practice is also important. Some practice is given in each exercise but more should be sourced from the text books now available, in follow up activities.

Transformations has long been an important Mathematical concept and is now an integral part of VCE Mathematical Methods. The course requires students to gain an understanding of how various transformations effect individual points on a plane, graphs of functions and their corresponding rules. A familiarity with the associated formal descriptive language and Mathematical notation is also required. While textbooks provide some insight, students often struggle to understand the form of presentation and many are dependent on their teachers for interpretation. With a few basic understandings and skills, students can explore these ideas themselves and grow in their appreciation of Mathematics. Your students may have had some exposure in junior secondary levels to transformations of geometric objects. If so, then they have already laid important groundwork. This workbook is intended for students from Year 10 upwards and covers reflections in the x and y-axes, translations parallel to the x and y-axes and dilations from the x and y-axes.

The guided exercises are designed to:

- Allow students to explore the effects of transformations on individual points , graphs of functions and their corresponding rules
- Encourage Mathematical thinking
- Teach the formal language, both worded and symbolic, that is used to describe transformations.

Students should be able to work through these exercises without much assistance. Solutions are provided and students should be encouraged to check their work as they go. There is assumed familiarity with

- Sketching graphs using substitution and plotting points
- Basic algebraic techniques including simplification, expansion and factorisation

Less experienced students may have difficulty understanding the significance of the form in which some answers are given. These students may only get to the first line of working but they should be able to follow and verify the truth of all steps in the given solution. Being able to rewrite answers in different forms is also a useful skill.

I haven't provided all possible variations of correct answers. The choice was generally made in favour of more elegant and common forms.

Example

Task	Given Solution	Advice
Given $f(x)=(x-1)^2+1$, the task is to find the image function g after a dilation from the Y-axis of factor 4.	$g(x)=(\frac{x}{4}-1)^2+1$	This is a straightforward substitution and should not pose a problem for most students at year 10 and above. Some students may then expand and simplify as follows: $g(x)=(\frac{x}{4}-1)^2+1$ $=\frac{x^2}{16}-\frac{x}{2}+1+1$ $=\frac{x^2}{16}-\frac{x}{2}+2$ This is fine but you could also show them this more elegant presentation: $\frac{1}{16}\left(x^2-8x+32\right)$
	$=\left(\frac{1}{4}\left(x-4\right)\right)^2+1$	The reason for this partial factorization will elude some students and may require explanation. However, they should be able to verify that it is correct by expanding the inner bracket and comparing with the previous step. Also, it is a common error to omit the outer bracket here and then forget to square the $\frac{1}{4}$ in the next step.
	$=\frac{1}{16}\left(x-4\right)^2+1$	This is turning point form and should be recognized by students above year 10.

The transformations in this booklet are applied only to functions. The nature of the chosen transformations is such that the images will also be functions, which ties in nicely with the function notation used throughout. Of course, the transformations presented can be applied to any relation.

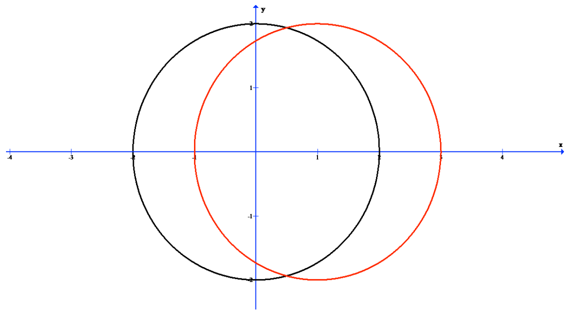
When your students have completed the booklet, you could look at transformations of relations that are not functions.

e.g. Consider the circle $x^2 + y^2 = 4$. Find the equation of the image after

a a translation of 1 unit in the positive x direction

We can't use function notation $f(x) \to f(x-1)$ to describe this transformation but the effect of the transformation on the graph and rule are familiar.

The image equation is:
$(x-1)^2 + y^2 = 4$.

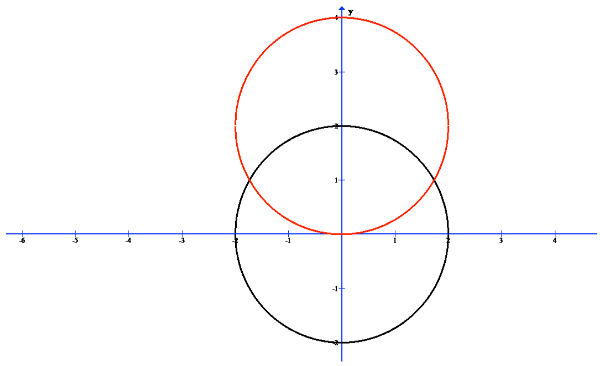


b a translation of 2 units in the positive y direction

Once again, we can't use function notation $f(x) \to f(x)+2$ to describe this transformation but the effect of the transformation on the graph and rule are familiar.

$$x^2 + y^2 = 4$$
$$\Leftrightarrow y = \pm\sqrt{4-x^2}$$

The image equation is:
$y = \pm\sqrt{4-x^2} + 2$ or
$x^2 + (y-2)^2 = 4$



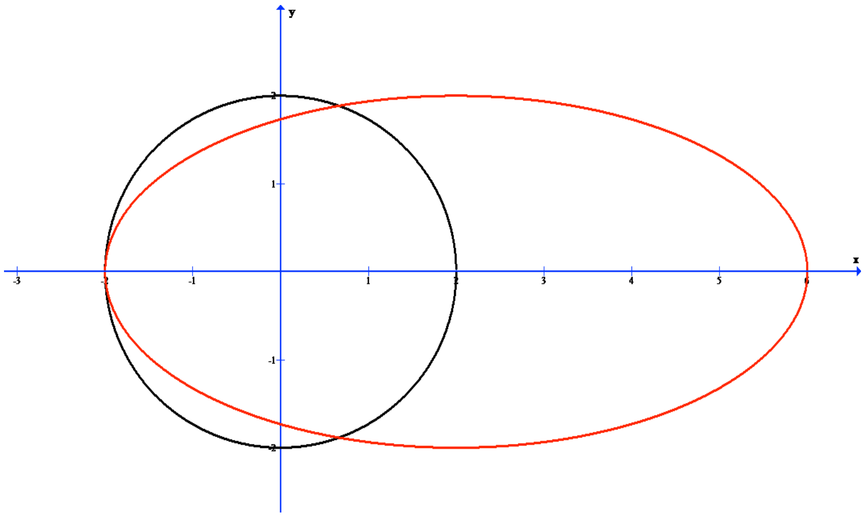
Some possible further extensions of this exercise:

- Consider dilations (parallel to the x axis/parallel to the y axis) of the given circle, finding the rule and noting changes to the shape of the graph.
- Consider the effect of combining transformations. Note that the order of transformations may produce a different result.

e.g. A translation of 1 to the right followed by dilation from the y axis of factor 2 produces:

$y^2 + \left(\frac{x}{2} - 1\right)^2 = 4$ or

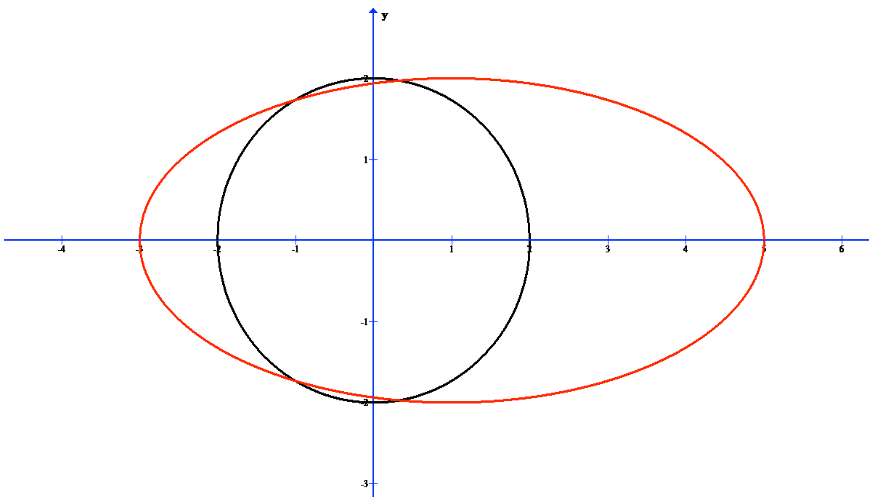
$4y^2 + (x-2)^2 = 16$



but a dilation from the y axis of factor 2 followed by a translation of 1 unit to the right produces:

$y^2 + \left(\frac{x-1}{2}\right)^2 = 4$ or

$4y^2 + (x-1)^2 = 16$



Note: Your students should be able to predict the image graphs without technology just by considering the effect of the transformations on a few well chosen points.

- Consider the relation $x^2 + y^2 = r^2$.

  The graph of this relation is a circle with centre $(0,0)$ and radius r.

  The relation undergoes a translation of h units to the right, followed by a translation of k units up. Will this transformation produce a change in the shape or size of the original circle? Why or why not?

  Show that the image equation is $(x-h)^2 + (y-k)^2 = r^2$.

  What are the coordinates of the centre of the image graph?

The following will be covered in subsequent workbooks:

- Combinations of transformations
- Reflections in the line $y = x$
- Inverse transformations
- Given the graph of an elementary*function $y = f(x)$, applying knowledge of transformations to then graph functions of the form $y = Af(n(x+b))+c$ where $A, n, b \text{ and } c \in R, \text{ and } A, n \neq 0$ **.
- Describing transformations with matrices
- Families of transformed functions

Teaching Mathematics to a diverse group at any level is challenging but can also be enormously rewarding. Mathematics, when taught well, will make sense to your students. Try to ask questions that encourage logical thought. When you can, help your students to make connections between old and new ideas. Give plenty of practice to strengthen your students' familiarity with new work and build confidence. Finally, and most importantly, enjoy yourself. Your enthusiasm is contagious!

Jana Kohout, 2016

*The VCAA study design for Mathematical Methods refers to "elementary functions of a single variable" and specifies "power functions, exponential functions, circular functions and polynomial functions".

** This form is taken directly from the VCAA study design, 2016.

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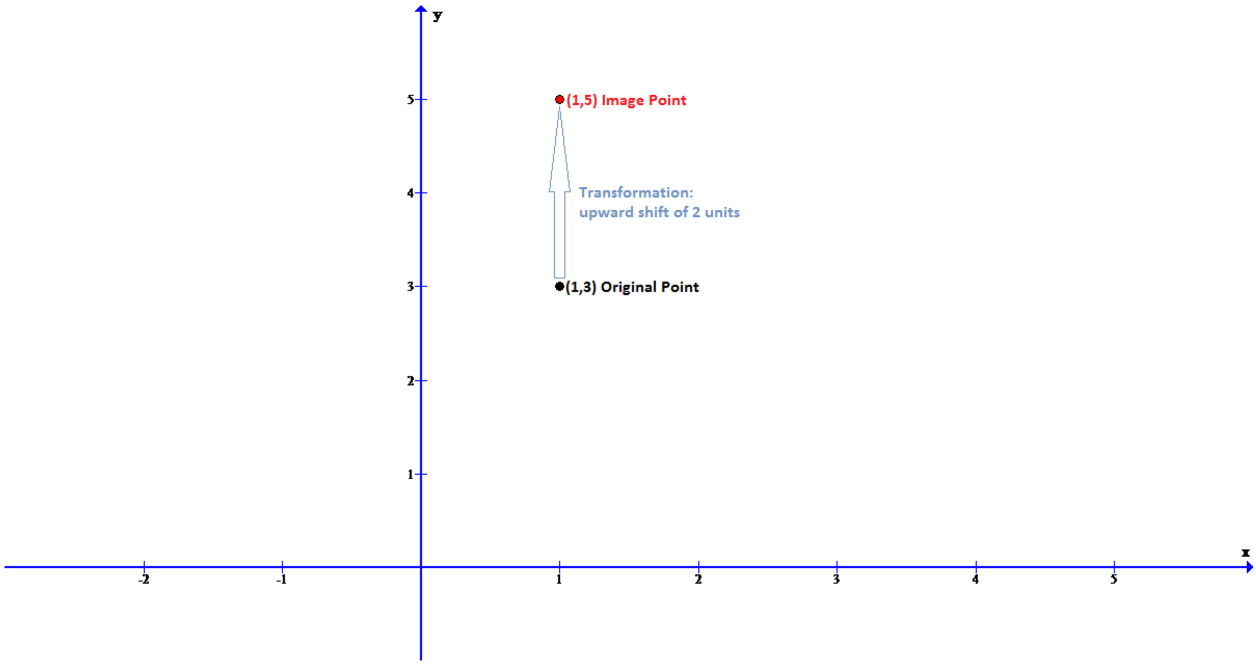
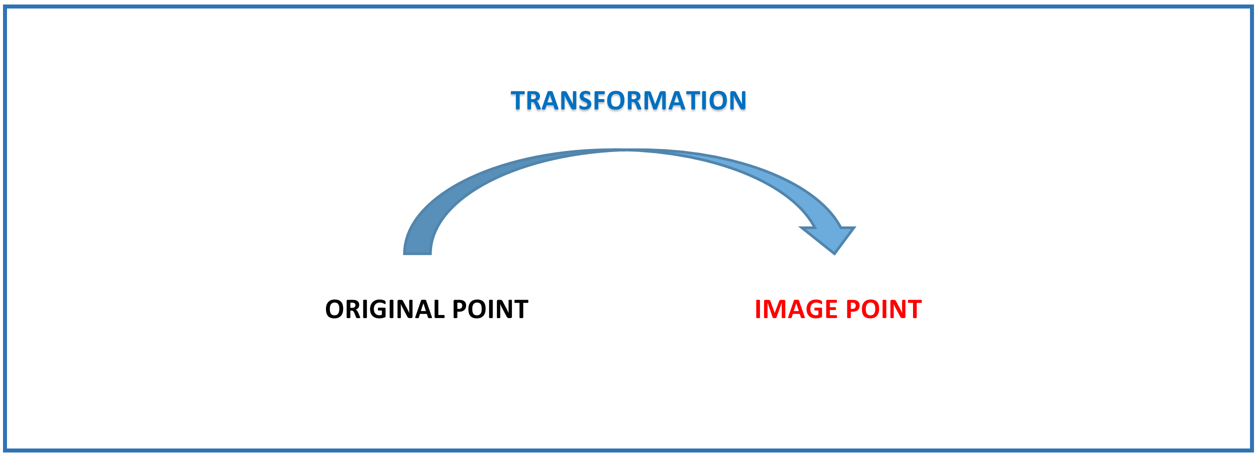
Exploring Transformations

When a point undergoes a transformation, it is shifted to a new position.

The point $(1,3)$ is transformed by adding 2 to its y-coordinate and becomes $(1,5)$. On a Cartesian plane the transformation is seen as a vertical shift upwards of the original point $(1,3)$. The new point $(1,5)$ is known as the image point.





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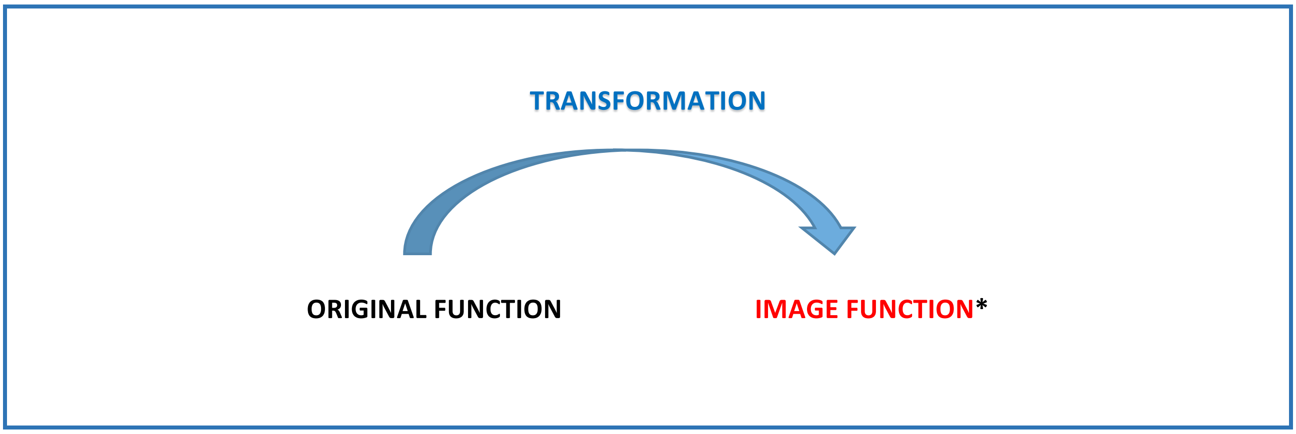
The graph of any function is just a collection of points. When a function undergoes a transformation, the points that make up the original graph are shifted to new positions following some fixed rule that is applied to all the points e.g. add 2 to all the x-coordinates or multiply all the y-coordinates by 3. Consequently there may be changes to the overall shape and/or position of the graph. There may also be changes to the appearance of the rule. The transformed function is known as the image function.*



Exercise 1 Reflection in the x-axis

1 The original function $f(x) = x^2$ is transformed into its image function, g following the rule "multiply all the y-coordinates by -1". Complete the table.

Points on f	$(-3\ ,\ 9)$ X -1	$(-2\ ,\ 4)$ X-1	$(-1\ ,\ \_\_\_)$ X-1	$(0\ ,\ \_\_\_)$ X-1	$(1\ ,\ \_\_\_)$ X-1	$(2\ ,\ \_\_\_)$ X-1	$(3\ ,\ \_\_\_)$ X-1	$(x\ ,\ \_\_\_)$ X-1
Points on g	$(-3, -9)$	$(-2, -4)$	$(-1\ ,\ \_\_\_)$	$(0\ ,\ \_\_\_)$	$(\_\_\_\ ,\ \_\_\_)$	$(\_\_\_\ ,\ \_\_\_)$	$(\_\_\_\ ,\ \_\_\_)$	$(x\ ,\ \_\_\_)$

* Note: The six transformations covered in this booklet, when applied to functions, result in images that are also functions. In fact, transformations can be applied to any relation, not just functions. Also, there are certain transformations which when applied to functions may not produce function images. This will be explored later. Right now, if you don't recall what a function is or you are unfamiliar with function notation, read appendix A on page 55.

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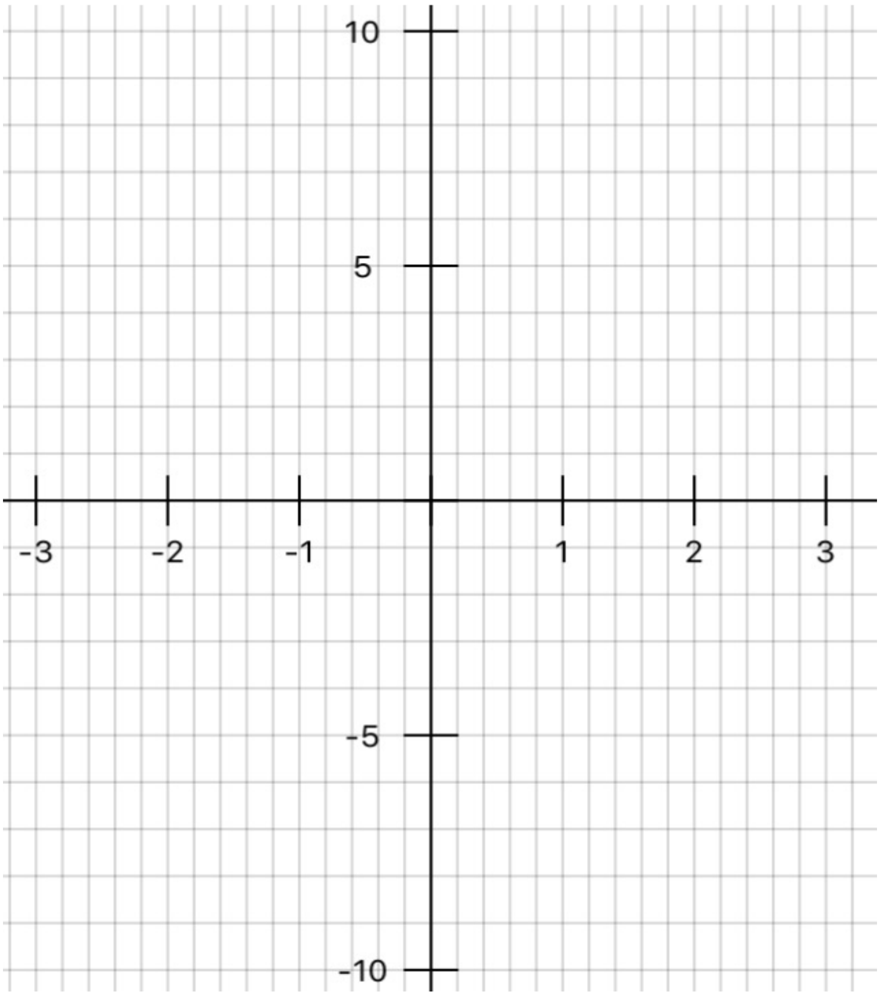
2 Using the table in question 1 to help, sketch the graphs of f and g on the same axes below.

Assume functions f and g have a domain of R.**



** Only seven values of x (-3,-2,-1, 0, 1, 2 and 3) have been chosen for the table in question 1. Functions f and g having a domain of R, simply means that the x values are not restricted to those in the table. x can be any number including any positive or negative number, zero, a fraction, surd or other irrational number like π , in fact any number on the number line.

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The formal language of transformations

Note that the graph of g is the mirror image of the graph of f with the x-axis acting as the mirror. This will always be the case when the y-coordinates of all points on the original graph are multiplied by -1. Here is a selection of phrases commonly used to describe this type of transformation:

"g is a mirror image of f in the x-axis"

"g is a reflection of f in the x-axis"

"The transformation is a reflection in the x-axis and f is mapped to g under this transformation."

Also, if g is a reflection of f in the x-axis then it is true to say that f is a reflection of g in the x-axis.

3 Complete the following for the above functions f and g:

original function: $f(x) =$ ______________

transformation: reflection in the ______________

image function: $g(x) =$ ______________

What does function g do?

Input Output

x---->--- g -->----??

Hint: Read appendix A and see the highlighted part of the table in question 1.

The image equation

The mirror image of $f(x)$ in the x-axis is $-f(x)$. You will see this formally written as

"$f(x) \to -f(x)$".

Say: "f of x maps to negative f of x"

This just means that the rule for the original function is multiplied by -1 to obtain the rule for the image.

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e.g. If $f(x) = x^3 - 3x^2 + x - 2$, then the reflection of f in the x-axis is

$$-f(x) = -(x^3 - 3x^2 + x - 2)$$
$$-f(x) = -x^3 + 3x^2 - x + 2$$

Why are the brackets important here?

Renaming the image as function g, we now have

$$g(x) = -x^3 + 3x^2 - x + 2$$

4 Which of the following function pairs are reflections of each other in the x-axis?

Decide by examining the equations only. Do not sketch the graphs.

If the answer is "No", what is the equation for the reflection of f in the x-axis?

	f	g	Are f and g reflections of each other in the x-axis?
a	$f(x) = x^3$	$g(x) = -x^3$	
b	$f(x) = x^2 + 1$	$g(x) = -(x^2 + 1)$	
c	$f(x) = x + 1$	$g(x) = -x + 1$	
d	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-x}$	

5 For each pair of functions in question 4, graph f and its correct image g on the same axes. A graphing calculator may be used, if needed.

Are your graphs clearly mirror images of each other in the x-axis?

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Exercise 2 Translations up and down

1a $f(x) = 2x$ is transformed into function g following the rule "add 1 to all the y-coordinates". Complete the table.

Points on f	$(-3, -6)$	$(-2, -4)$	$(-1, \_\_)$	$(0, \_\_)$	$(1, \_\_)$	$(2, \_\_)$	$(3, \_\_)$	$(x, \_\_)$
	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$	$\downarrow +1$
Points on g	$(-3, -5)$	$(-2, -3)$	$(-1, \_\_)$	$(0, \_\_)$	$(\_\_, \_\_)$	$(\_\_, \_\_)$	$(\_\_, \_\_)$	$(x, \_\_)$

b $f(x) = 2x$ is transformed into function *h* following the rule "subtract 3 from all the y-coordinates". Complete the table.

Points on f	$(-3, -6)$	$(-2, -4)$	$(-1, \_\_)$	$(0, \_\_)$	$(1, \_\_)$	$(2, \_\_)$	$(3, \_\_)$	$(x, \_\_)$
	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$	$\downarrow -3$
Points on h	$(-3, -9)$	$(-2, -7)$	$(-1, \_\_)$	$(0, \_\_)$	$(\_\_, \_\_)$	$(\_\_, \_\_)$	$(\_\_, \_\_)$	$(x, \_\_)$

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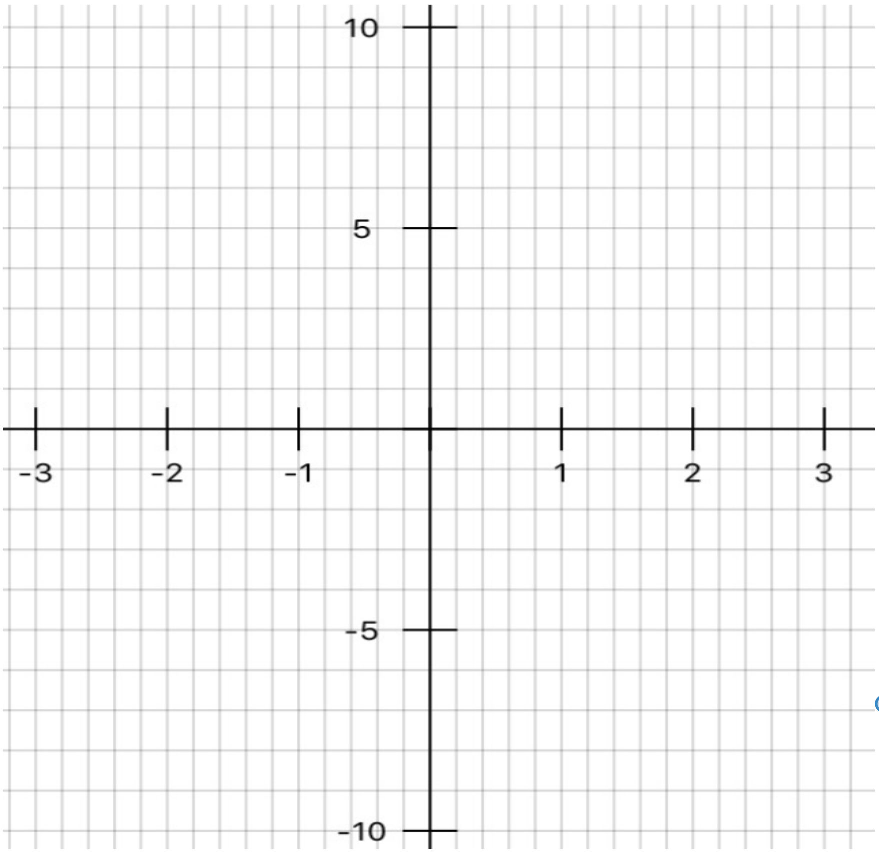
2 Using the tables from question 1 to help, sketch the graphs of f, g and h on the same axes below. Assume functions f, g and h have a domain of R.



The formal language of transformations

Note that the graph of g is just the graph of f with all points shifted up by 1 unit. Similarly, the graph of h is just the graph of f with all points shifted down 3 units. Here is a selection of phrases commonly used to describe these types of transformations:

"g is the image of f after a translation of 1 unit up"

"g is the image of f after a translation of 1 unit in the positive direction of the y -axis "

"h is the image of f after a translation of 3 units down"

"h is the image of f after a translation of 3 units in the negative direction of the y -axis"

"The transformation is a translation of 1 unit in the positive direction of the y-axis and f is mapped to g under this transformation."

"The transformation is a translation of 3 units in the negative direction of the y-axis and f is mapped to h under this transformation."

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3 Complete the following for the above functions f, g and h:

a original function: $f(x) =$ ______________

transformation: ___________ of __ unit in the __________ direction of the ________

image: $g(x) =$ ______________

Hint: See the highlighted part of the table in question 1a.

b original function: $f(x) =$ ______________

transformation: ___________ of __ units in the _________ direction of the _________

image: $h(x) =$ ______________

Hint: See the highlighted part of the table in question 1b.

The image equation

Note A translation of *c* units up (where *c* is some positive constant) results in the original rule having *c* added to it. A formal way to express this idea is $f(x) \to f(x) + c$. Similarly, a translation of *c* units down can be formally written as $f(x) \to f(x) - c$.

Say: "f of x maps to f of x plus c"

How would you say this?

e.g. The image of $f(x) = 3x^2$ after a translation of 5 units in the positive direction of the y-axis

is

$$f(x) + 5 = 3x^2 + 5$$

Renaming the image as function g, we now have

$$g(x) = 3x^2 + 5$$

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e.g. The image of $f(x) = (x+1)^3$ after a translation of 4 units in the negative direction of the y-axis is

$$f(x) - 4 = (x+1)^3 - 4$$

$$\text{or } f(x) - 4 = x^3 + 3x^2 + 3x + 1 - 4$$

$$= x^3 + 3x^2 + 3x - 3$$

Renaming the image as function g, we now have

$$g(x) = (x+1)^3 - 4$$

$$\text{or } g(x) = x^3 + 3x^2 + 3x - 3$$

4 g is the image of f after a translation of 2 units in the positive direction of the y-axis.

h is the image of f after a translation of π units in the negative direction of the y-axis.

Complete the following table by finding the rules for g and h.

	f	g	h
a	$f(x) = x^3$		
b	$f(x) = -3x + 1$		
c	$f(x) = (x-1)^2$		
d	$f(x) = 2\sqrt{x}$		

5 Check your answers to each part of question 4 by graphing f, g and h on the same axes. A graphing calculator may be used, if needed.

In each case, can you see that g is the same shape as f but shifted up by 2 units?

Similarly, can you see that h is the same shape as f but shifted down by a little over 3 units?

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Exercise 3 Vertical dilations

1a $f(x)=x^2+2$ is transformed into function *g* following the rule "multiply all the y- coordinates by 2". Complete the table.

Points on f	$(-2\ ,\ 6)$ X2	$(-1\ ,\ \_\_\_)$ X2	$(\ 0\ ,\ \_\_\_)$ X2	$(\ 1\ ,\ \_\_\_)$ X2	$(\ 2\ ,\ \_\_\_)$ X2	$(\ x\ ,\ \_\_\_\_\_\_\_\_\_)$ X2
Points on g	$(-2\ ,\ 12)$	$(-1\ ,\ \_\_\_)$	$(\ 0\ ,\ \_\_\_)$	$(\ \_\_\_\ ,\ \_\_\_)$	$(\ \_\_\_\ ,\ \_\_\_)$	$(\ x\ ,\ \_\_\_\_\_\_\_\_\_)$

b $f(x)=x^2+2$ is transformed into function *h* following the rule "multiply all the y-coordinates by $\frac{1}{2}$". Complete the table.

Points on f	$(-2\ ,\ 6)$ $\times\frac{1}{2}$	$(-1\ ,\ \_\_\_)$ $\times\frac{1}{2}$	$(\ 0\ ,\ \_\_\_)$ $\times\frac{1}{2}$	$(\ 1\ ,\ \_\_\_)$ $\times\frac{1}{2}$	$(\ 2\ ,\ \_\_\_)$ $\times\frac{1}{2}$	$(\ x\ ,\ \_\_\_\_\_\_\_\_\_)$ $\times\frac{1}{2}$
Points on h	$(-2\ ,\ 3\ )$	$(-1\ ,\ \_\_\_)$	$(\ 0\ ,\ \_\_\_)$	$(\ \_\_\_\ ,\ \_\_\_)$	$(\ \_\_\_\ ,\ \_\_\_)$	$(\ x\ ,\ \_\_\_\_\_\_\_\_\_)$

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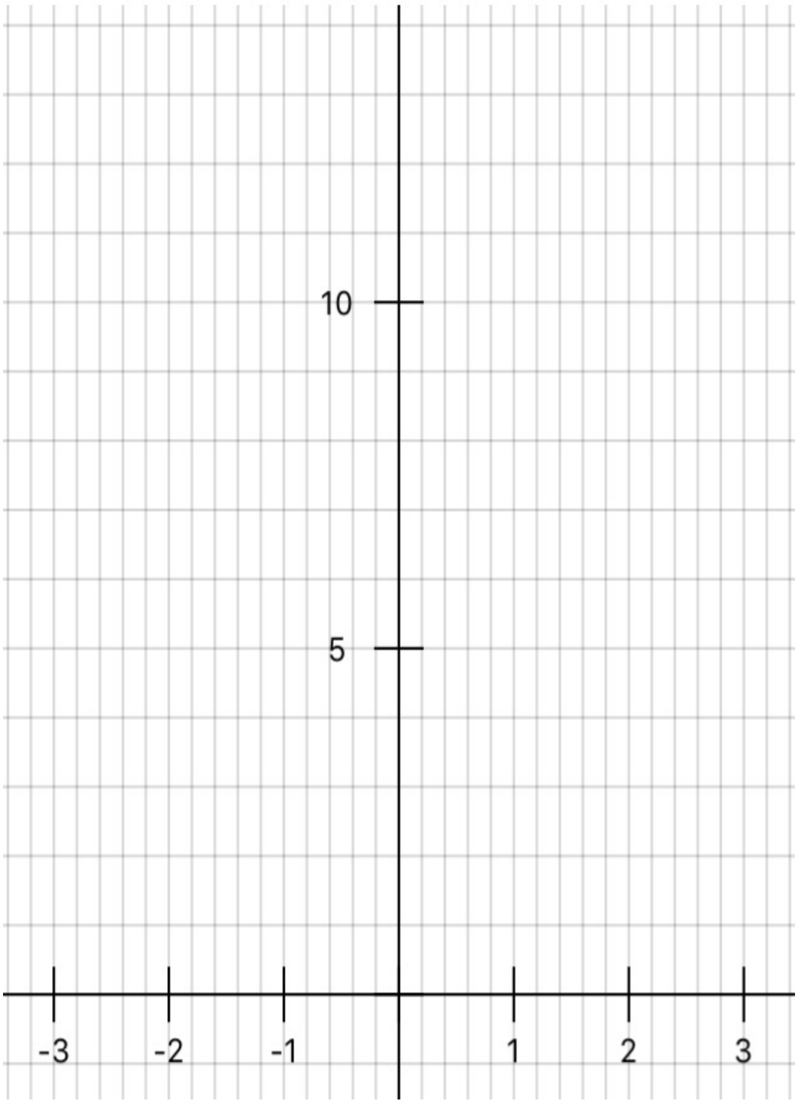
2 Using the tables from question 1 to help, sketch the graphs of f, g and h on the same axes below. Assume functions f, g and h have a domain of R.



The formal language of transformations

Note that the transformation of f into g causes a "vertical stretching" of f so the image graph appears thinner than the original. This will always be the case when the y-coordinates of all points on the original graph are multiplied by a constant larger than 1. Vertical stretching can also be thought of as a stretching away from the x-axis.

Also note that the transformation of f into h causes a "vertical squashing" of f so the image graph appears wider than the original. This will always be the case when the y-coordinates of all points on the original graph are multiplied by a positive constant smaller than 1. Vertical squashing can also be thought of as a squashing towards the x-axis.

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Here is a selection of phrases commonly used to describe these types of transformations:

"g is the image of f after a dilation from the x-axis of factor 2"

"h is the image of f after a dilation from the x-axis of factor $\frac{1}{2}$"

"g is the image of f after a dilation parallel to the y-axis of factor 2"

"h is the image of f after a dilation parallel to the y-axis of factor $\frac{1}{2}$"

"The transformation is a dilation from the x-axis of factor 2 and f is mapped to g under this transformation."

"The transformation is a dilation from the x-axis of factor $\frac{1}{2}$ and f is mapped to h under this transformation."

3 Complete the following for the above functions f, g and h:

a original function: $f(x) =$ ______________

transformation: dilation from the ___________ of factor ______

image function: $g(x) =$ ______________

b original function: $f(x) =$ ______________

transformation: dilation parallel to the _________ of factor ______

image function: $h(x) =$ ______________

The image equation

Note that the image functions g and h above are just multiples of the original function f.

g is $\mathbf{2}f(x) = \mathbf{2}(x^2 + 2)$ and h is $\mathbf{\frac{1}{2}}f(x) = \mathbf{\frac{1}{2}}(x^2 + 2)$

In general, a dilation from the x-axis of factor k results in the original rule being multiplied by k. A formal way to express this idea is $f(x) \to kf(x)$.

Say: "f of x maps to k times f of x"

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eg If $f(x) = 2x^2 - x + 1$ and f undergoes a dilation from the x-axis of factor 4, then the image is

$$4f(x) = 4(2x^2 - x + 1)$$
$$= 8x^2 - 4x + 4$$

Why are the brackets important here?

Renaming the image as function g, we now have

$$g(x) = 8x^2 - 4x + 4$$

4 g is the image of f after a dilation from the x- axis of factor 3

h is the image of f after a dilation parallel to the y-axis of factor $\frac{3}{4}$

Complete the following table by finding the rules for g and h.

	f	g	h
a	$f(x) = x^3$		
b	$f(x) = -x + 1$		
c	$f(x) = (x-1)^2 + 4$		
d	$f(x) = 2\sqrt{x}$		

5 Check your answers to each part of question 4 by graphing f, g and h on the same axes. A graphing calculator may be used, if needed.

In each case, can you see the vertical stretching of f that produces g?

Similarly, can you see that h is a vertically squashed version of f?

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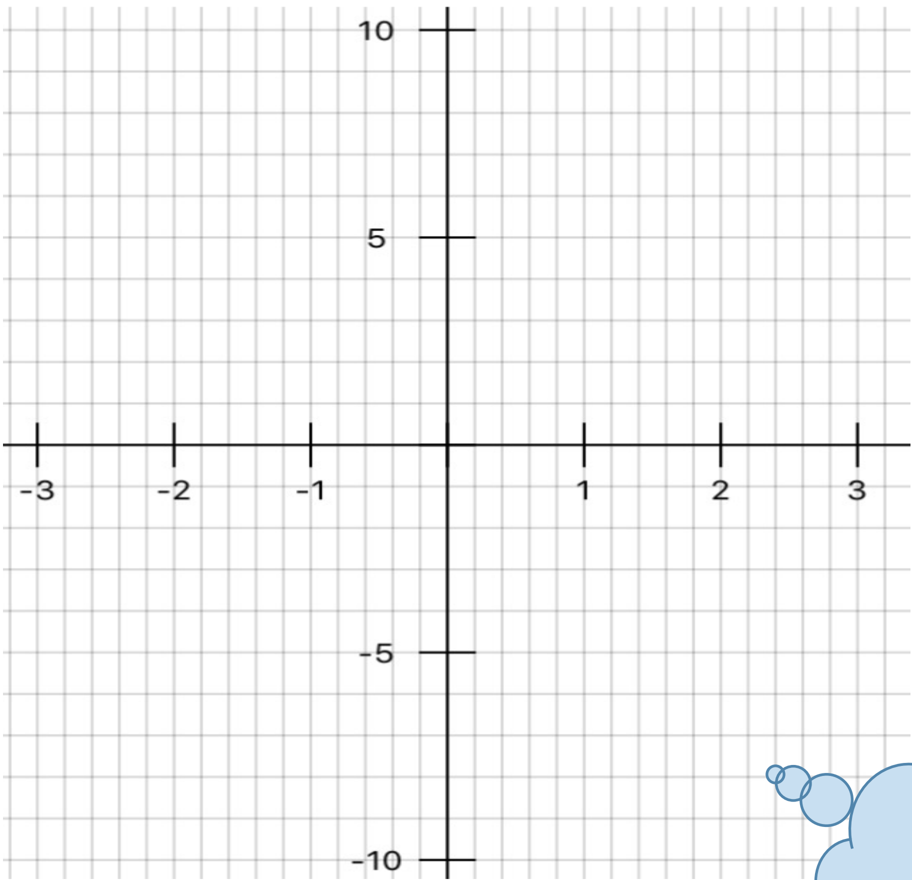
Exercise 4 Reflection in the y-axis

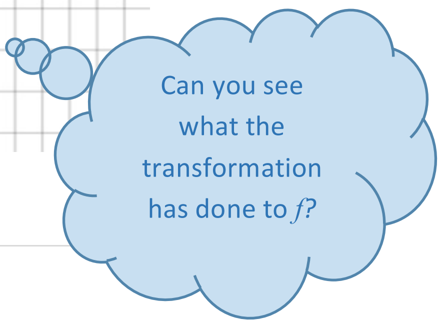
1 $f(x)=x^3+1$ is transformed into function g following the rule "multiply all the x-coordinates by -1". Complete the table.

Points on f	$(2,9)$	$(\_\_,\_\_)$	$(\_\_,\_\_)$	$(\_\_,\_\_)$	$(\_\_,\_\_)$	$(\_\_,\_\_\_\_)$
	X-1	X-1	X-1	X-1	X-1	X-1
Points on g	$(-2,9)$	$(-1,2)$	$(0,\_\_)$	$(1,\_\_)$	$(2,\_\_)$	$(x,\_\_\_\_)$

2 Using the table from question 1, sketch the graphs of f and g on the same axes below.

Assume functions f and g both have a domain of R.





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The formal language of transformations

Note that the graphs of f and g are mirror images of each other with the y-axis acting as the mirror. This will always be the case when the x-coordinates of all points on the original graph are multiplied by -1. Here is a selection of phrases commonly used to describe this type of transformation:

"g is a mirror image of f in the y-axis"

"g is a reflection of f in the y-axis"

"The transformation is a reflection in the y-axis and f is mapped to g under this transformation."

Also, if g is a reflection of f in the y-axis then it is true to say that f is a reflection of g in the y-axis.

3 Complete the following for the above functions f and g:

original function: $f(x) =$ ______________

transformation: reflection in the _____________

image: $g(x) =$ _____________

The image equation

The mirror image of $f(x)$ in the y-axis is $f(-x)$. You will see this formally written as

"$f(x) \to f(-x)$".

Say: "f of x maps to f *of negative* x"

This simply means that to obtain the image equation in this case, replace every instance of "x" in the original equation with "$(-x)$".

eg If $f(x) = x^2 - x$, then the mirror image of f in the y-axis is

$$f(-x) = (-x)^2 - (-x)$$
$$= x^2 + x$$

Notice the brackets around every instance of "$-x$" in this image equation. It would be incorrect to write

$$f(-x) = -x^2 - -x$$
$$= x^2 + x$$

Why?

Renaming the image as function g, we now have

$$g(x) = x^2 + x$$

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eg If $f(x)=x^3-3x^2+x-2$, then the mirror image of f in the y-axis is

$$\begin{aligned} f(-x) &= (-x)^3-3(-x)^2+(-x)-2 \\ &= -x^3-3x^2-x-2 \end{aligned}$$

Renaming the image as function g, we now have

$$g(x)=-x^3-3x^2-x-2$$

4 Which of the following function pairs are reflections of each other in the y-axis?

Decide by examining the equations only. Do <u>not</u> sketch the graphs.

If the answer is "No", what is the equation for the reflection of f in the y-axis?

	f	g	Are f and g reflections of each other in the y- axis?
a	$f(x)=x+2$	$g(x)=-x+2$	
b	$f(x)=x^2+1$	$g(x)=-x^2+1$	
c	$f(x)=\sqrt{x}$	$g(x)=\sqrt{-x}$	
d	$f(x)=x(x+1)^2$	$g(x)=-x(x-1)^2$	

5 For each pair of functions in question 4, graph f and its <u>correct image</u> g on the same axes. A graphing calculator may be used, if needed.

Are your graphs clearly mirror images of each other in the y-axis?

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Exercise 5 Translations left and right

1a $f(x) = x^2 - 1$ is transformed into function g following the rule "add 1 to all the x-coordinates". Complete the table.

Points on f	$(-2,3)$ $\downarrow$ +1	$(\_\_,\_\_)$ $\downarrow$ +1	$(\_\_,\_\_)$ $\downarrow$ +1	$(\_\_,\_\_)$ $\downarrow$ +1	$(\_\_,\_\_)$ $\downarrow$ +1	$(\_\_\_\_,\_\_\_\_)$ $\downarrow$ +1
Points on g	$(-1,3)$	$(0,\ 0)$	$(1\ ,-1)$	$(2\ ,\_\_)$	$(3\ ,\_\_)$	$(x\ ,\_\_\_\_\_\_)$

b $f(x) = x^2 - 1$ is transformed into function h following the rule "subtract 2 from all the

x- coordinates". Complete the table.

Points on f	$(-2,3)$ $\downarrow$ -2	$(\_\_,\_\_)$ $\downarrow$ -2	$(\_\_,\_\_)$ $\downarrow$ -2	$(\_\_,\_\_)$ $\downarrow$ -2	$(\_\_,\_\_)$ $\downarrow$ -2	$(\_\_\_\_,\_\_\_\_)$ $\downarrow$ -2
Points on h	$(-4,3)$	$(-3,\ 0)$	$(-2,\ -1)$	$(-1\ ,\_\_)$	$(0\ ,\_\_)$	$(x\ ,\_\_\_\_\_\_)$

2 Using the tables from question 1, sketch the graphs of f, g and h on the same axes below.

Assume functions f, g and h have a domain of R.

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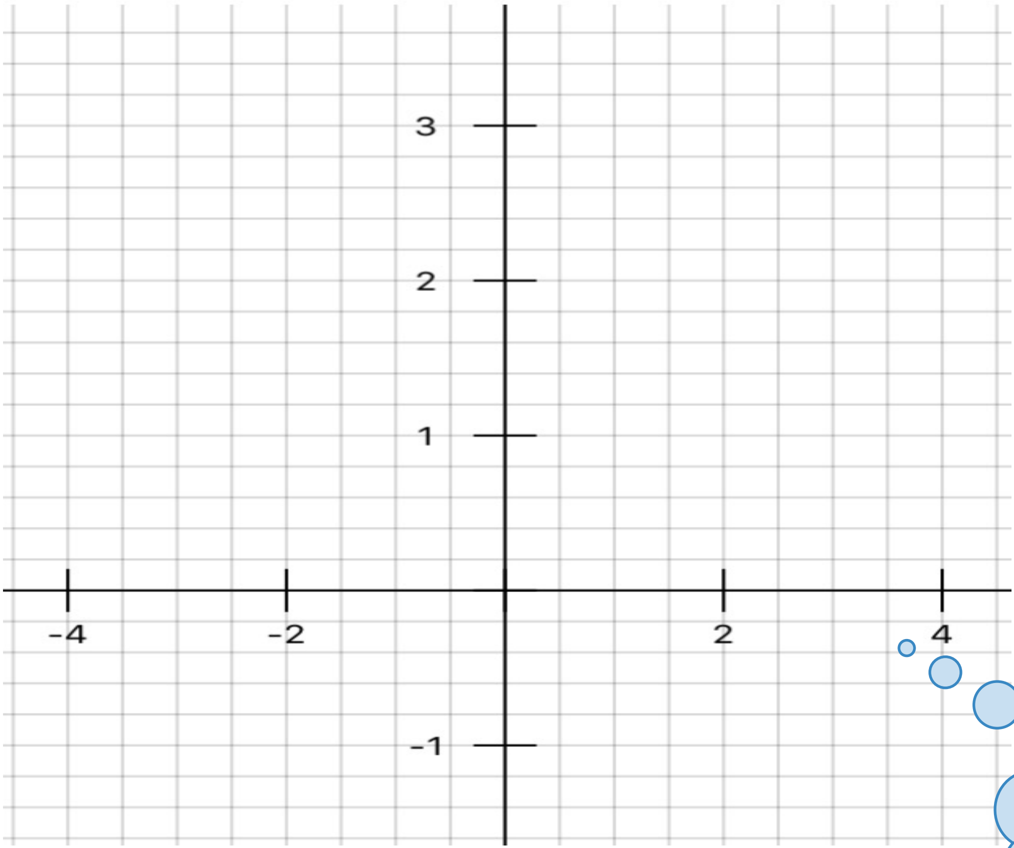

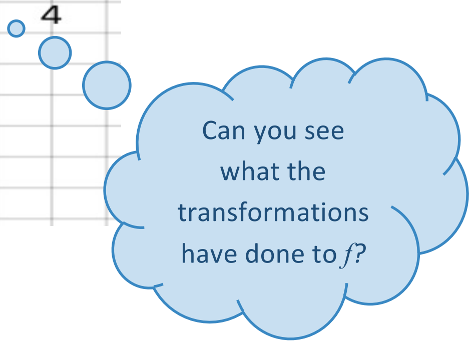

The formal language of transformations

Note that the graph of g is just the graph of f with all points shifted right by 1 unit. Similarly, the graph of h is just the graph of f with all points shifted left 2 units. Here is a selection of phrases commonly used to describe these types of transformations:

"g is the image of f after a translation of 1 unit to the right"

"g is the image of f after a translation of 1 unit in the positive direction of the x-axis"

"h is the image of f after a translation of 2 units to the left"

"h is the image of f after a translation of 2 units in the negative direction of the x-axis"

"The transformation is a translation of 1 unit in the positive direction of the x-axis and f is mapped to g under this transformation."

"The transformation is a translation of 2 units in the negative direction of the x-axis and f is mapped to h under this transformation"

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3 Complete the following for the above functions f, g and h:

a original function: $f(x) =$ _____________

transformation: translation of __ unit in the _________________

image: $g(x) =$ _____________

b original function: $f(x) =$ _____________

transformation: translation of __ units in the _________________

image: $h(x) =$ _____________

The image equation

If function $f(x)$ undergoes a translation of c units to the right (where c is some positive constant) then its image is $f(\boldsymbol{x-c})$.

You will see this formally written as "$f(x) \to f(x-c)$".

Say: "*f* of x maps to *f of* x *minus* c"

This simply means that to obtain the image equation in this case, replace every instance of "x" in the original equation with "$(x-c)$".

Similarly, if function $f(x)$ undergoes a translation of c units to the left then its image is $f(\boldsymbol{x+c})$. You will see this formally written as "$f(x) \to f(x+c)$".

How would you say this?

This simply means that to obtain the image equation in this case, replace every instance of "x" in the original equation with "$(x+c)$".

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e.g. If $f(x)=x^2-3x+1$, and f undergoes a translation of 3 units to the right then its image is

$$\begin{aligned} f(\boldsymbol{x-3}) &= (\boldsymbol{x-3})^2-3(\boldsymbol{x-3})+1 \\ &= x^2-6x+9-3x+9+1 \\ &= x^2-9x+19 \end{aligned}$$

Renaming the image as function g, we now have

$$g(x)=x^2-9x+19$$

e.g. The image of $f(x)=(2x+1)^3$ after a translation of 4 units in the negative direction of the x-axis is

$$\begin{aligned} f(\boldsymbol{x+4}) &= (2(\boldsymbol{x+4})+1)^3 \\ &= (2x+8+1)^3 \\ &= (2x+9)^3 \\ &= 8x^3+108x^2+486x+729 \end{aligned}$$

Renaming the image as function g, we now have

$$g(x)=(2x+9)^3$$

or $g(x)=8x^3+108x^2+486x+729$

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e is an irrational number,

approximately equal to 2.71828

$e \approx 2.71828$

4 g is the image of f after a translation of 2 units to the right.

h is the image of f after a translation of e units in the negative direction of the x-axis.

Complete the following table by finding the rules for g and h.

	f	g	h
a	$f(x) = x^3 - 1$		
b	$f(x) = -3x^2 + x$		
c	$f(x) = (x+1)^2$		
d	$f(x) = 2\sqrt{1-x}$		

5 Check your answers to each part of question 4 by graphing f, g and h on the same axes.

A graphing calculator may be used, if needed.

In each case, can you see that g is the same shape as f but shifted right by 2 units?

Similarly, can you see that h is the same shape as f but shifted left by about 2.7 units?

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Exercise 6 Horizontal dilations

1a $f(x)=x^3$ is transformed into function g following the rule "multiply all the x-coordinates by 2".
Complete the table.

Points on f	$(-2,-8)$ X2	$(\_\_,\_\_)$ X2	$(\_\_,\_\_)$ X2	$(\_\_,\_\_)$ X2	$(\_\_,\_\_)$ X2	$(\_\_\_\_,\_\_\_\_)$ X2
Points on g	$(-4,-8)$	$(-2,-1)$	$(0,0)$	$(2,\_\_)$	$(4,\_\_)$	$(x\quad,\_\_\_\_)$

b $f(x)=x^3$ is transformed into function *h* following the rule "multiply all the x-coordinates by $\frac{1}{2}$.

Complete the table.

Points on f	$(-2,-8)$ x$\frac{1}{2}$	$(\_\_,\_\_)$ x$\frac{1}{2}$	$(\_\_,\_\_)$ x$\frac{1}{2}$	$(\_\_,\_\_)$ x$\frac{1}{2}$	$(\_\_,\_\_)$ x$\frac{1}{2}$	$(\_\_\_\_,\_\_\_\_)$ x$\frac{1}{2}$
Points on h	$(-1,-8)$	$(-\frac{1}{2},-1)$	$(0,0)$	$(\frac{1}{2},\_\_)$	$(1,\_\_)$	$(x\quad,\_\_\_\_)$

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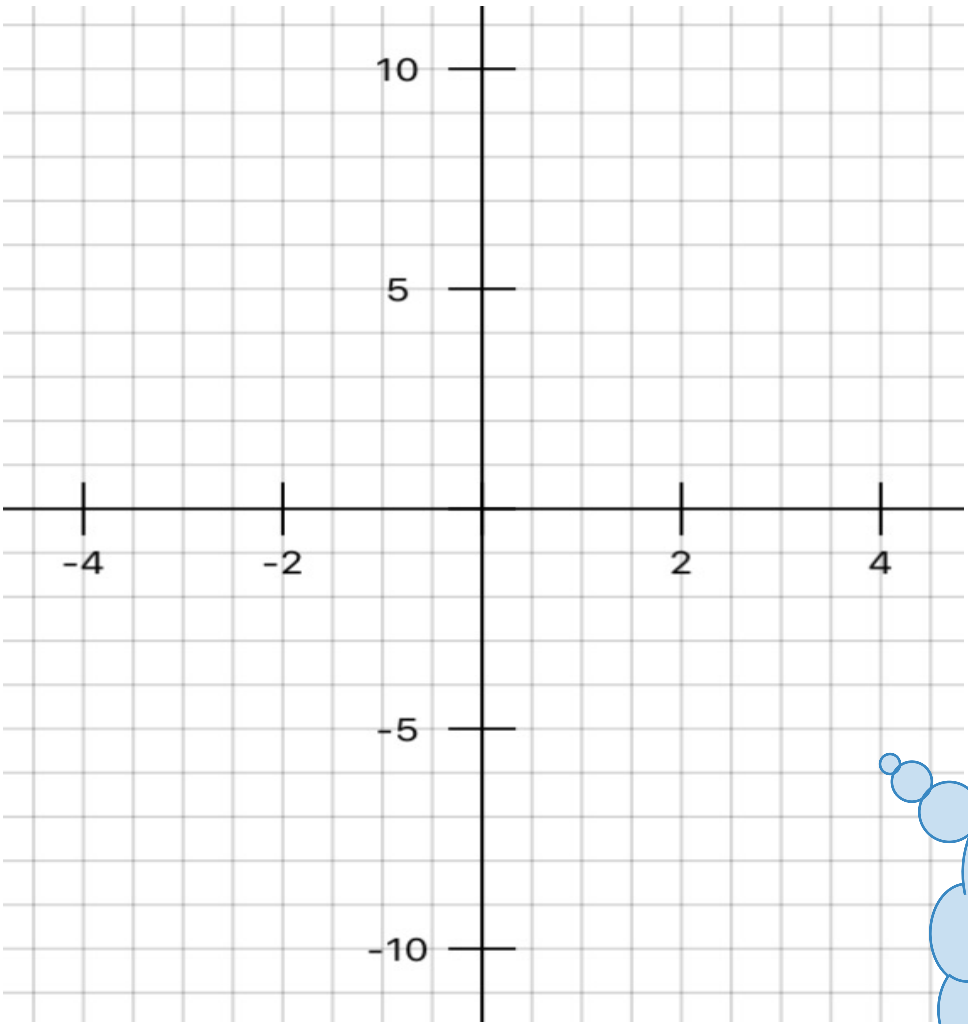
2 Using the tables from question 1, sketch the graphs of f, g and h on the same axes below.

Assume functions f, g and h have a domain of R.



Can you see what the transformations have done to f?

The formal language of transformations

Note that the transformation of f into g causes a “horizontal stretching” of the original graph so the image graph appears wider than the original. This will always be the case when the x-coordinates of all points on the original graph are multiplied by a constant larger than 1. Horizontal stretching can also be thought of as stretching away from the y-axis.

Also note that the transformation of f into h causes a “horizontal squashing” of f so the image graph appears thinner than the original. This will always be the case when the x-coordinates of all points on the

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original graph are multiplied by a positive constant smaller than 1. Horizontal squashing can also be thought of as squashing towards the y-axis.

Here is a selection of phrases commonly used to describe these types of transformations:

"g is the image of f after a dilation from the y-axis of factor 2"

"h is the image of f after a dilation from the y-axis of factor $\frac{1}{2}$"

"g is the image of f after a dilation parallel to the x-axis of factor 2"

"h is the image of f after a dilation parallel to the x-axis of factor $\frac{1}{2}$"

"The transformation is a dilation from the y-axis of factor 2 and f is mapped to g under this transformation."

"The transformation is a dilation from the y-axis of factor $\frac{1}{2}$ and f is mapped to h under this transformation."

3 Complete the following for the above functions f, g and h:

a original function: $f(x)=$ ____________

transformation: dilation from the __________ of factor ____

image: $g(x)=$ ____________

b original function: $f(x)=$ ____________

transformation: dilation parallel to the _________ of factor _____

image: $h(x)=$ ____________

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The image equation

If function $f(x)$ undergoes a dilation from the y-axis of factor k then its image is $f\left(\frac{x}{k}\right)$.

You will see this formally written as "$f(x) \to f\left(\frac{x}{k}\right)$".

How would you say this?

This simply means that to obtain the image equation in this case, replace every instance of "x" in the original equation with "$\left(\frac{x}{k}\right)$".

eg1 If $f(x) = 2x^2 - 3x$ undergoes a dilation from the y-axis of factor 3 then its image is

$$f\left(\frac{x}{3}\right) = 2\left(\frac{x}{3}\right)^2 - 3\left(\frac{x}{3}\right)$$
$$= \frac{2x^2}{9} - x$$

What's another way of saying this?

Renaming the image as function g, we now have

$$g(x) = \frac{2x^2}{9} - x$$

eg2 If $f(x) = 2x^2 - 3x$ undergoes a dilation from the y-axis of factor $\frac{1}{3}$ then its image is

$$f\left(\frac{x}{1/3}\right) = f(3x)$$
$$= 2(3x)^2 - 3(3x)$$
$$= 18x^2 - 9x$$

$$\frac{x}{1/3} = x \div \frac{1}{3}$$
$$= \frac{x}{1} \div \frac{1}{3}$$
$$= \frac{x}{1} \times \frac{3}{1}$$
$$= 3x$$

Renaming the image as function g, we now have

$$g(x) = 18x^2 - 9x$$

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4 g is the image of f after a dilation from the y-axis of factor 4

h is the image of f after a dilation parallel to the x-axis of factor $\dfrac{2}{3}$

Complete the following table by finding the rules for g and h.

	f	g	h
a	$f(x) = -x + 1$		
b	$f(x) = x^3$		
c	$f(x) = (x-1)^2 + 1$		
d	$f(x) = 2\sqrt{x}$		

5 Check your answers to each part of question 4 by graphing f, g and h on the same axes.

A graphing calculator may be used, if needed.

In each case, can you see the horizontal stretching of f that produces g?

Similarly, can you see that h is a horizontally squashed version of f?

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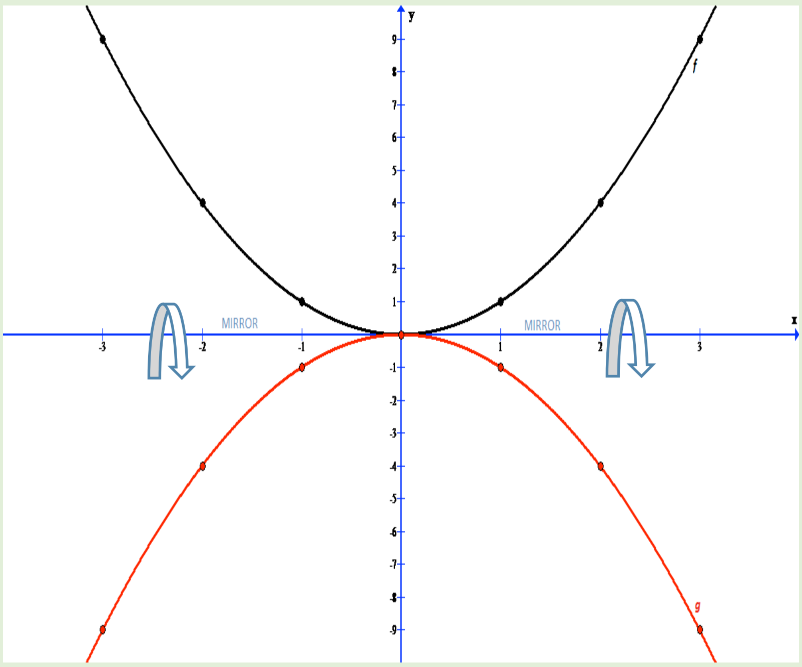
SOLUTIONS

Exercise 1 __Reflection in the x-axis__

1

Points on f	$(-3\ ,\ 9)$	$(-2\ ,\ 4)$	$(-1,1)$	$(0,0)$	$(1,1)$	$(2,4)$	$(3,9)$	(x,x^2)
Points on g	$(-3,-9)$	$(-2,-4)$	$(-1,-1)$	$(0,0)$	$(1,-1)$	$(2,-4)$	$(3,-9)$	$(x,-x^2)$

2



Multiplying the y-coordinates by -1 has produced a reflection of f.

g is the mirror image of f with the x-axis acting as the mirror.

3 original function: $f(x)=x^2$

transformation: reflection in the x-axis

image: $g(x)=-x^2$

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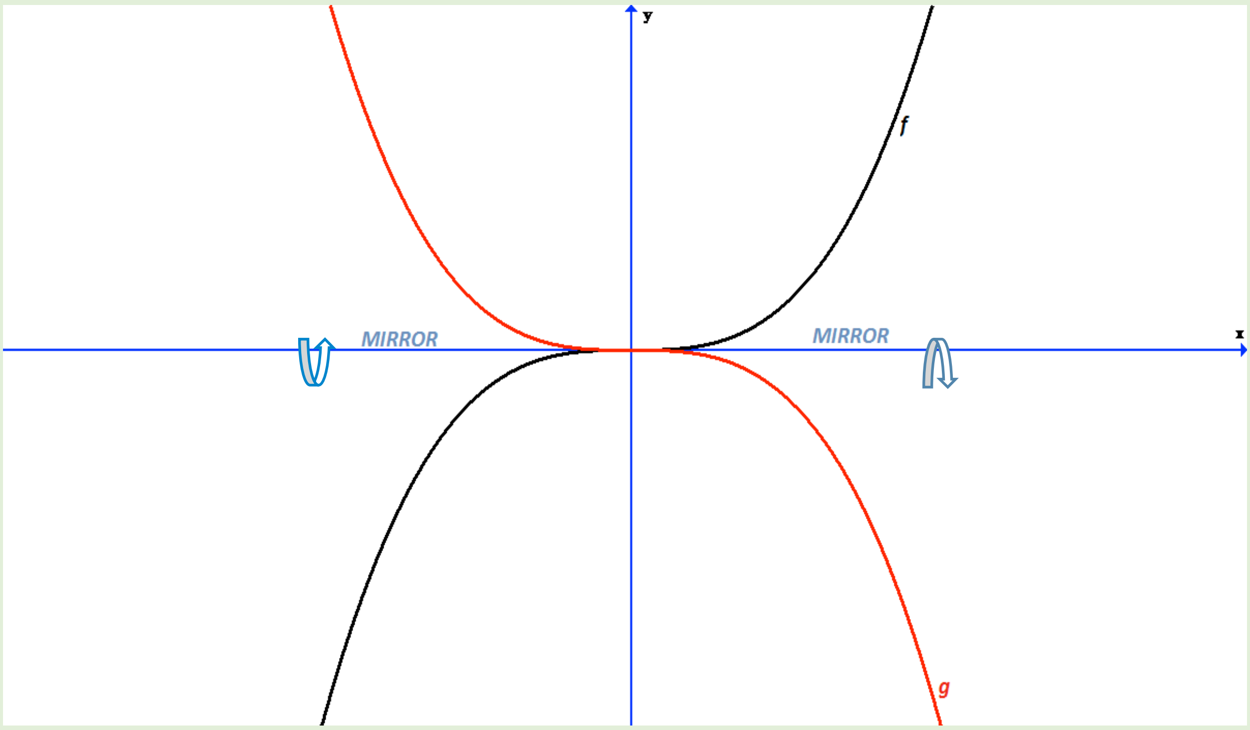
4

	f	g	Are f and g reflections of each other in the x-axis?
a	$f(x)=x^3$	$g(x)=-x^3$	YES
b	$f(x)=x^2+1$	$g(x)=-(x^2+1)$	YES
c	$f(x)=x+1$	$g(x)=-x+1$	NO $g(x)=-(x+1)$ or $g(x)=-x-1$
d	$f(x)=\sqrt{x}$	$g(x)=\sqrt{-x}$	NO $g(x)=-\sqrt{x}$

5

a



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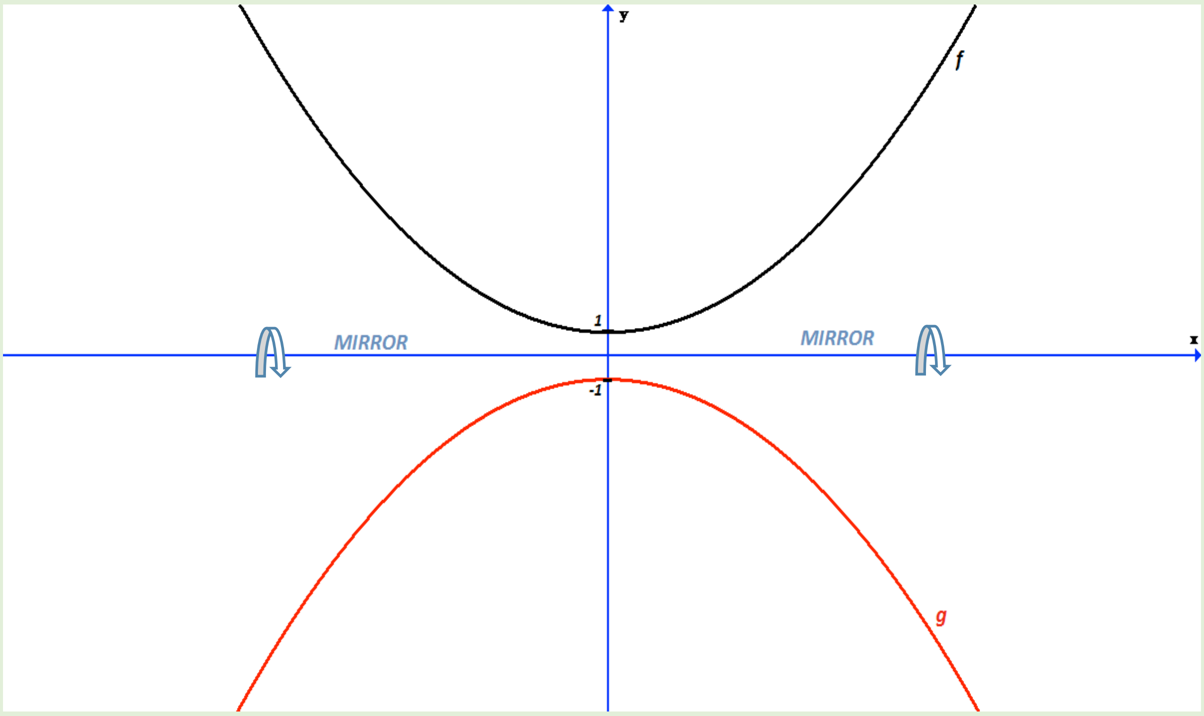
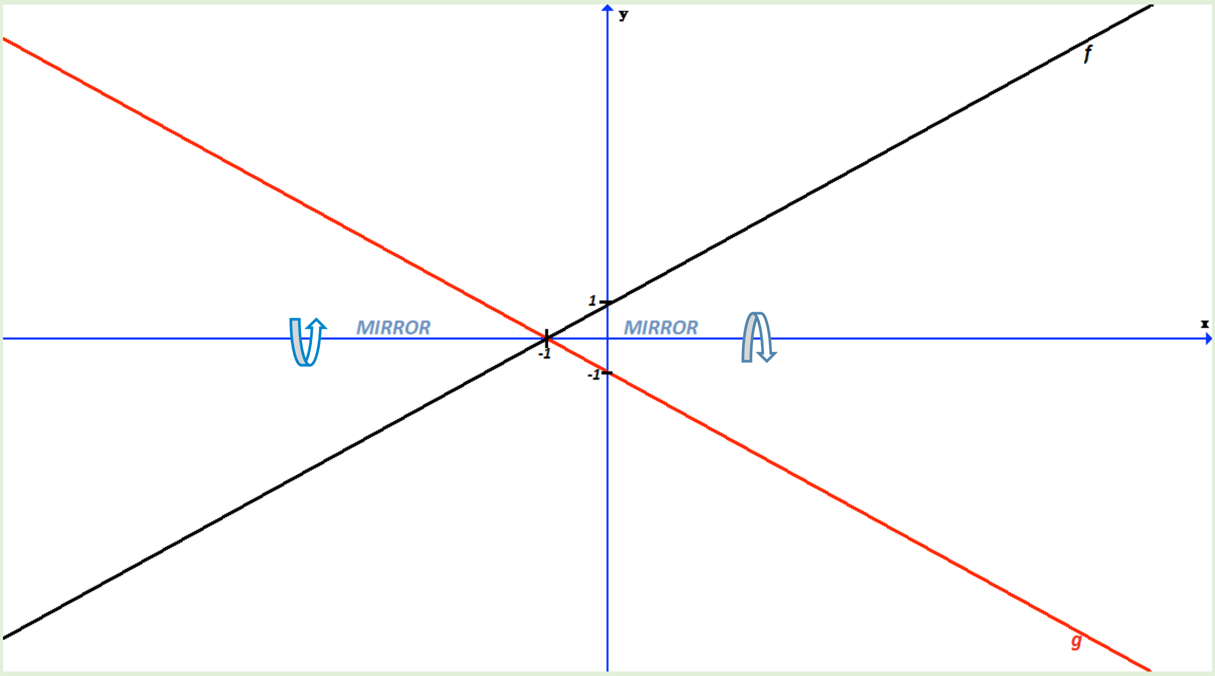
b



c



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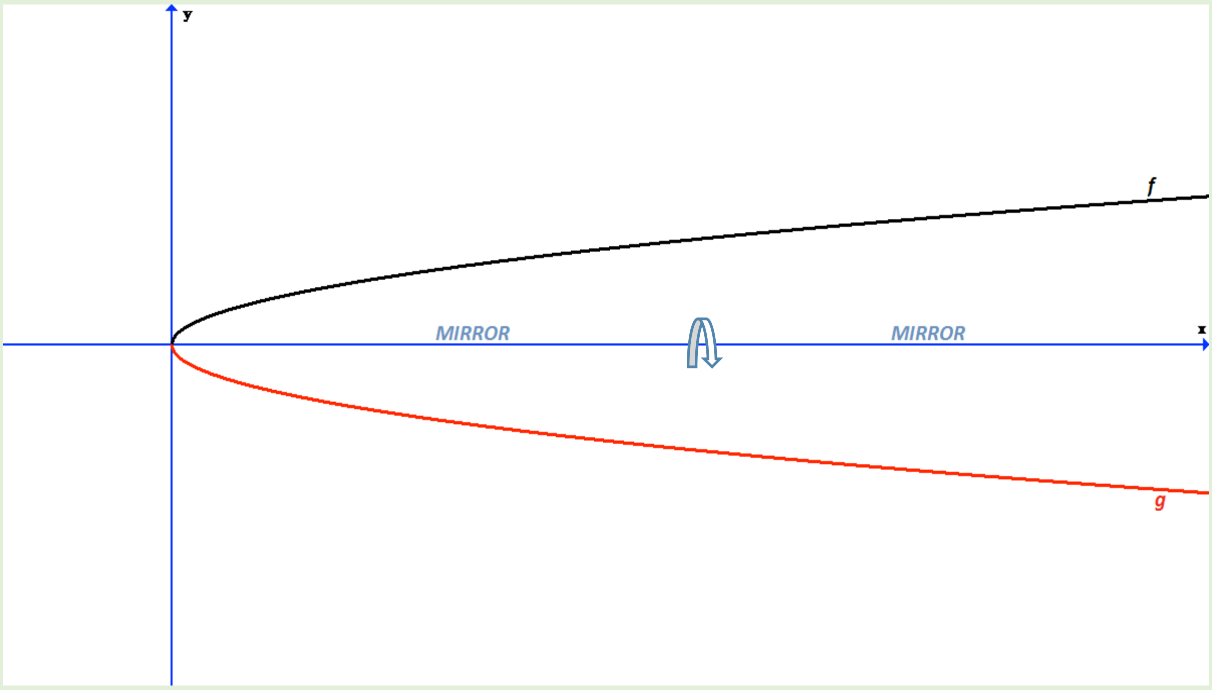
d



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Exercise 2 <u>**Translations up and down**</u>

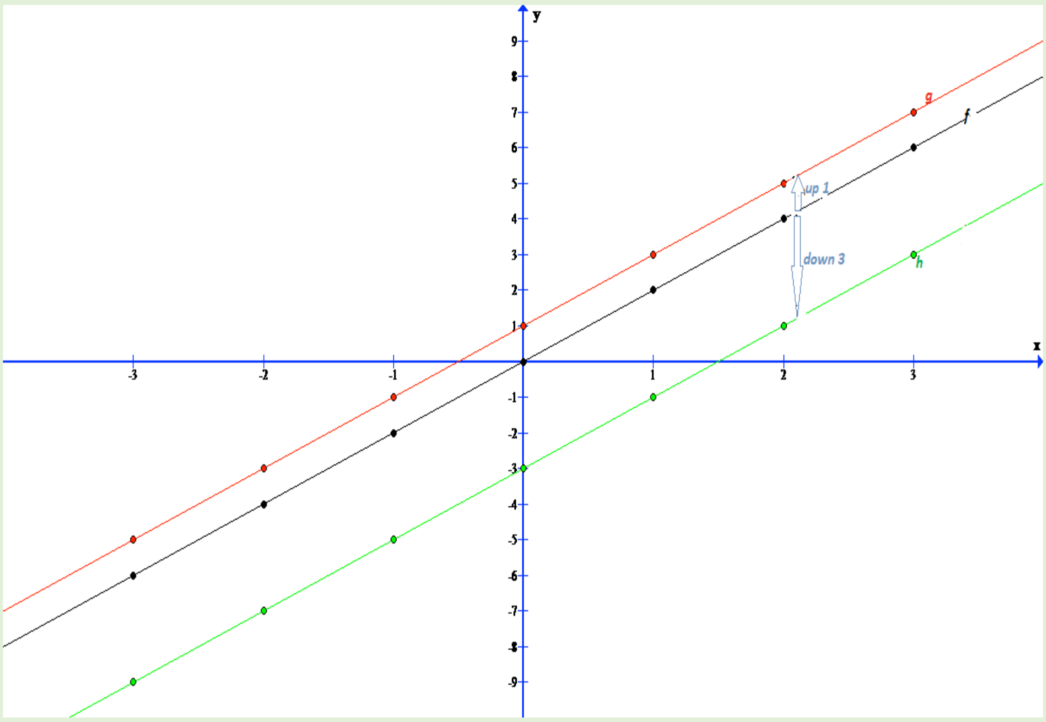
1a

Points on f	$(-3,-6)$	$(-2,-4)$	$(-1,-2)$	$(0,0)$	$(1,2)$	$(2,4)$	$(3,6)$	$(x,2x)$
Points on g	$(-3,-5)$	$(-2\ ,-3)$	$(-1,-1)$	$(0,1)$	$(1,3)$	$(2,5)$	$(3,7)$	$(x,2x+1)$

b

Points on f	$(-3,-6)$	$(-2,-4)$	$(-1,-2)$	$(0,0)$	$(1,2)$	$(2,4)$	$(3,6)$	$(x,2x)$
Points on h	$(-3,-9)$	$(-2,-7)$	$(-1,-5)$	$(0,-3)$	$(1,-1)$	$(2,1)$	$(3,3)$	$(x,2x-3)$

2



Adding 1 to the y-coordinates has shifted f by one unit up to make g.

Subtracting 3 from the y-coordinates has shifted f by three units down to make h.

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3

a original function: $f(x)=2x$ transformation: translation of 1 unit in the positive direction of the y-axis Image: $g(x)=2x+1$	**b** original function: $f(x)=2x$ transformation: translation of 3 units in the negative direction of the y-axis Image: $h(x)=2x-3$

4

	f	g	h
a	$f(x)=x^3$	$g(x)=x^3+2$	$h(x)=x^3-\pi$
b	$f(x)=-3x+1$	$g(x)=-3x+3$	$h(x)=-3x+1-\pi$
c	$f(x)=(x-1)^2$	$g(x)=(x-1)^2+2$ or $g(x)=x^2-2x+3$	$h(x)=(x-1)^2-\pi$
d	$f(x)=2\sqrt{x}$	$g(x)=2\sqrt{x}+2$	$h(x)=2\sqrt{x}-\pi$

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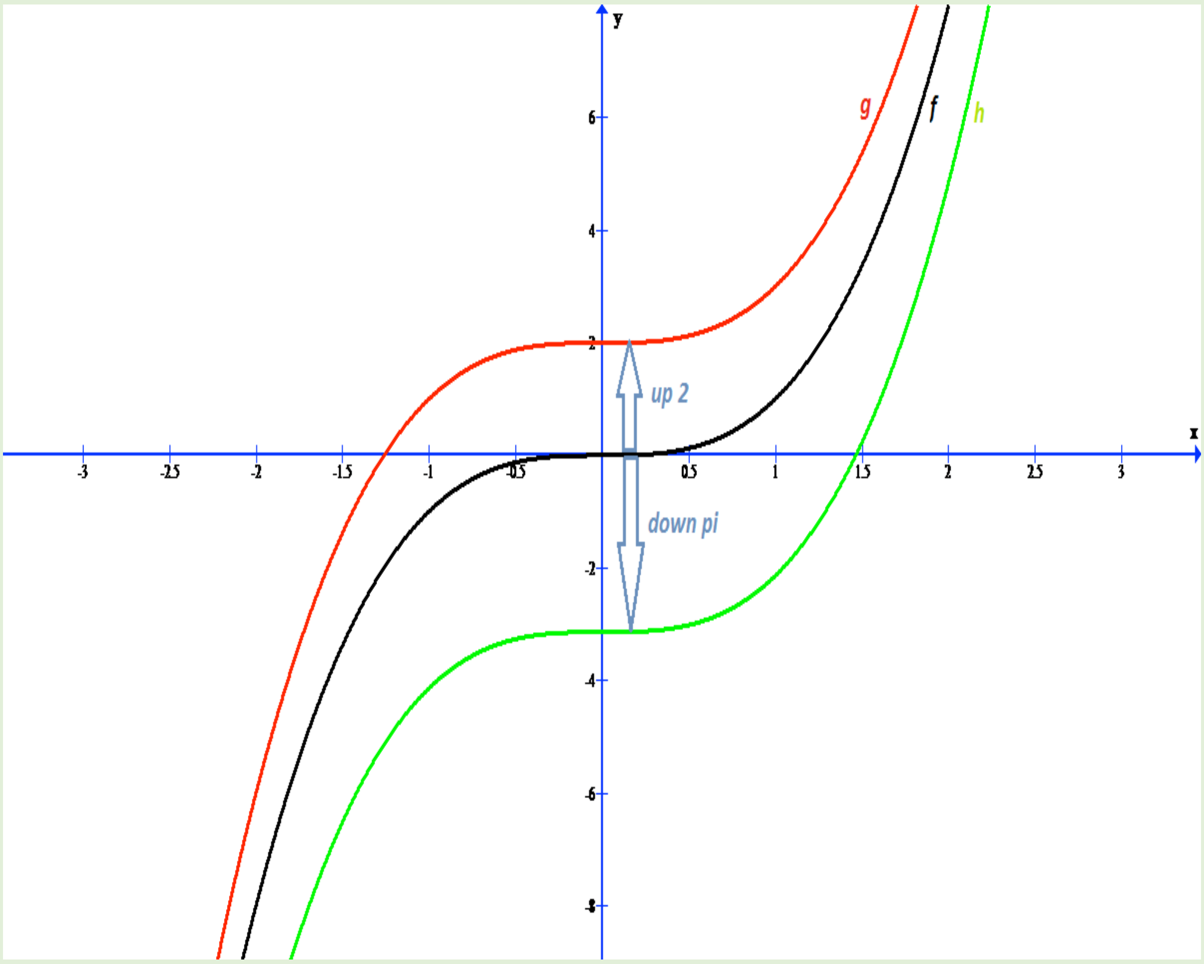
5

a



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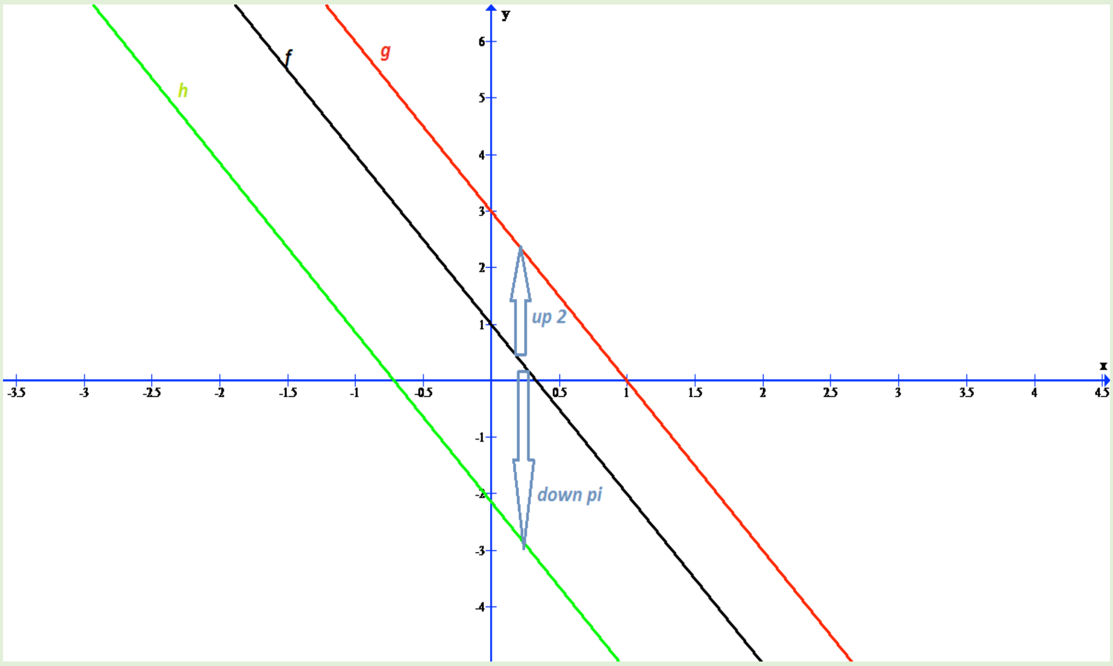
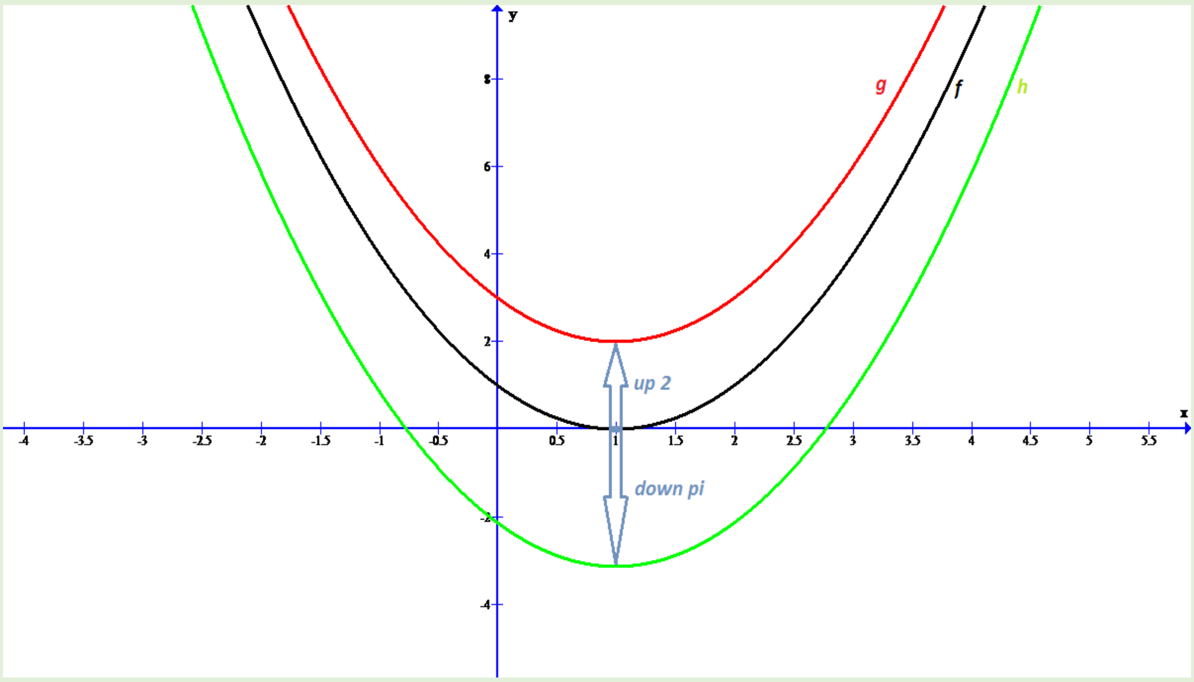
b



c



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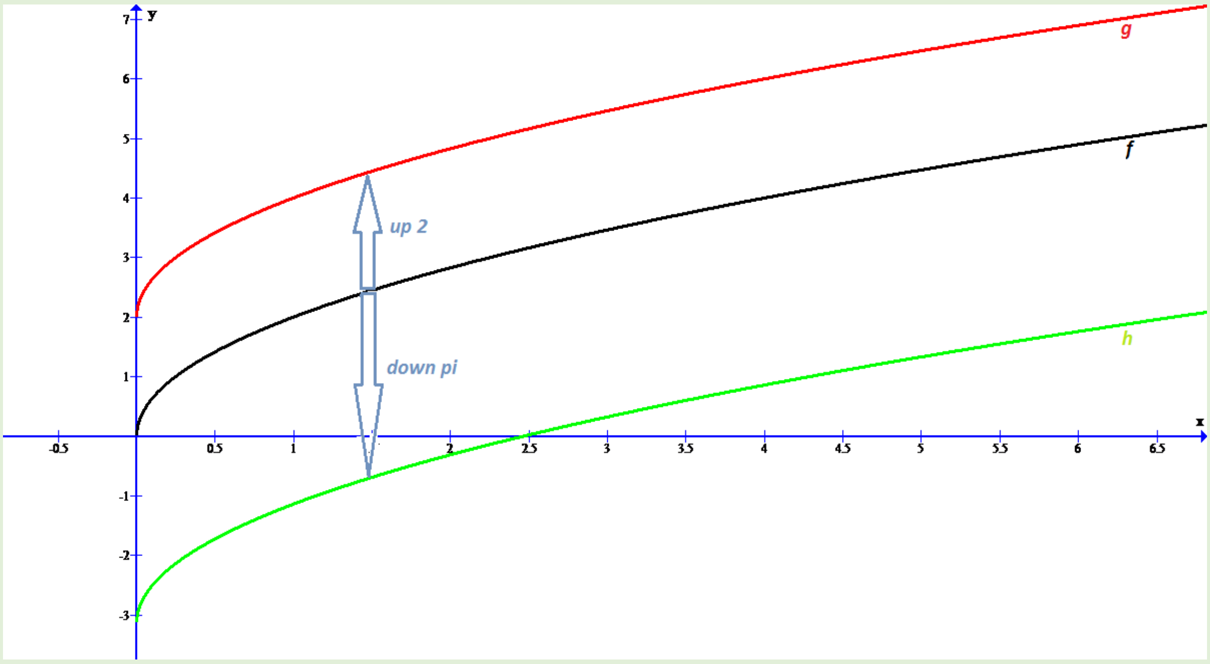
d



Exercise 3 __Vertical dilations__

1a

Points on f	$(-2,6)$	$(-1,3)$	$(0,2)$	$(1,3)$	$(2,6)$	(x,x^2+2)
Points on g	$(-2,12)$	$(-1,6)$	$(0,4)$	$(1,6)$	$(2,12)$	$(x,2(x^2+2))$ $=(x,2x^2+4)$

b

Points on f	$(-2,6)$	$(-1,3)$	$(0,2)$	$(1,3)$	$(2,6)$	(x,x^2+2)
Points on h	$(-2,3)$	$\left(-1,\frac{3}{2}\right)$ or $\left(-1,1\frac{1}{2}\right)$	$(0,1)$	$\left(1,\frac{3}{2}\right)$ or $\left(1,1\frac{1}{2}\right)$	$(2,3)$	$\left(x,\frac{1}{2}(x^2+2)\right)$ $=\left(x,\frac{x^2}{2}+1\right)$

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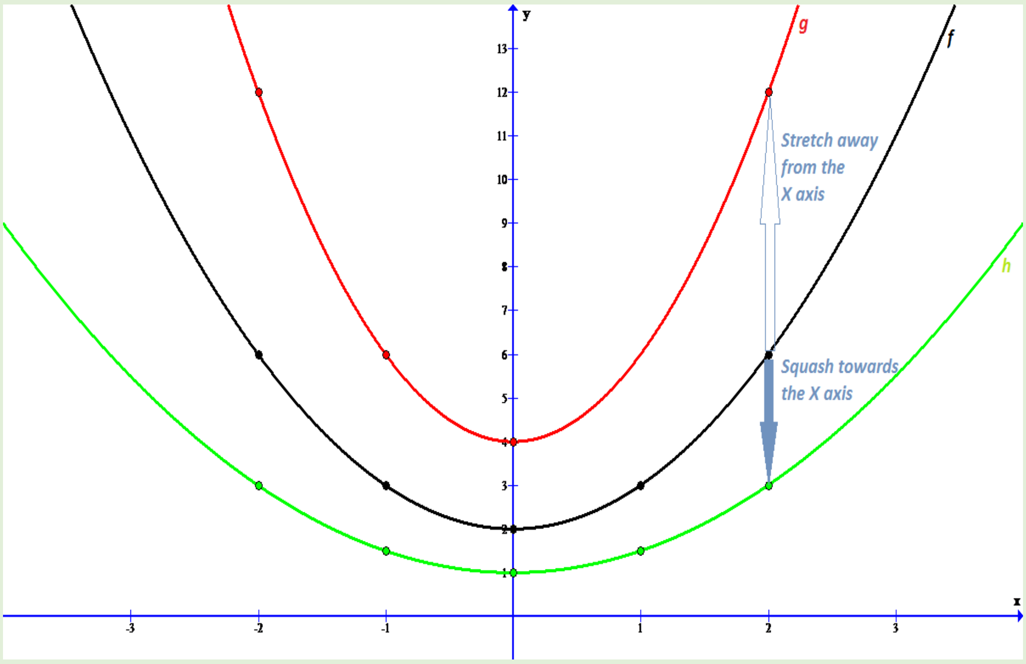
2



Multiplying the y-coordinates by 2 has stretched f vertically, away from the x-axis to make g.

Multiplying the y-coordinates by one half has squashed f vertically, towards the x-axis to make h.

3

a original function: $f(x)=x^2+2$ Transformation: dilation from the x -axis of factor 2 Image: $g(x)=2(x^2+2)$ or $g(x)=2x^2+4$	**b** original function: $f(x)=x^2+2$ Transformation: dilation parallel to the y- axis of factor $\frac{1}{2}$ Image: $h(x)=\frac{1}{2}(x^2+2)$ or $h(x)=\frac{x^2}{2}+1$

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4

	f	g	h
a	$f(x)=x^3$	$g(x)=3x^3$	$h(x)=\frac{3}{4}x^3$ or $h(x)=\frac{3x^3}{4}$
b	$f(x)=-x+1$	$g(x)=3(-x+1)$ $=-3x+3$	$h(x)=\frac{3}{4}(-x+1)$ $=-\frac{3}{4}x+\frac{3}{4}$ or $h(x)=-\frac{3x}{4}+\frac{3}{4}$
c	$f(x)=(x-1)^2+4$	$g(x)=3\left[(x-1)^2+4\right]$ $=3(x-1)^2+12$ or $g(x)=3(x^2-2x+1)+12$ $=3x^2-6x+15$	$h(x)=\frac{3}{4}\left[(x-1)^2+4\right]$ $=\frac{3}{4}(x-1)^2+3$ or $h(x)=\frac{3}{4}(x^2-2x+1)+3$ $=\frac{3}{4}x^2-\frac{3}{2}x+3\frac{3}{4}$
d	$f(x)=2\sqrt{x}$	$g(x)=3\left(2\sqrt{x}\right)$ $=6\sqrt{x}$	$h(x)=\frac{3}{4}\left(2\sqrt{x}\right)$ $=\frac{3}{2}\sqrt{x}$

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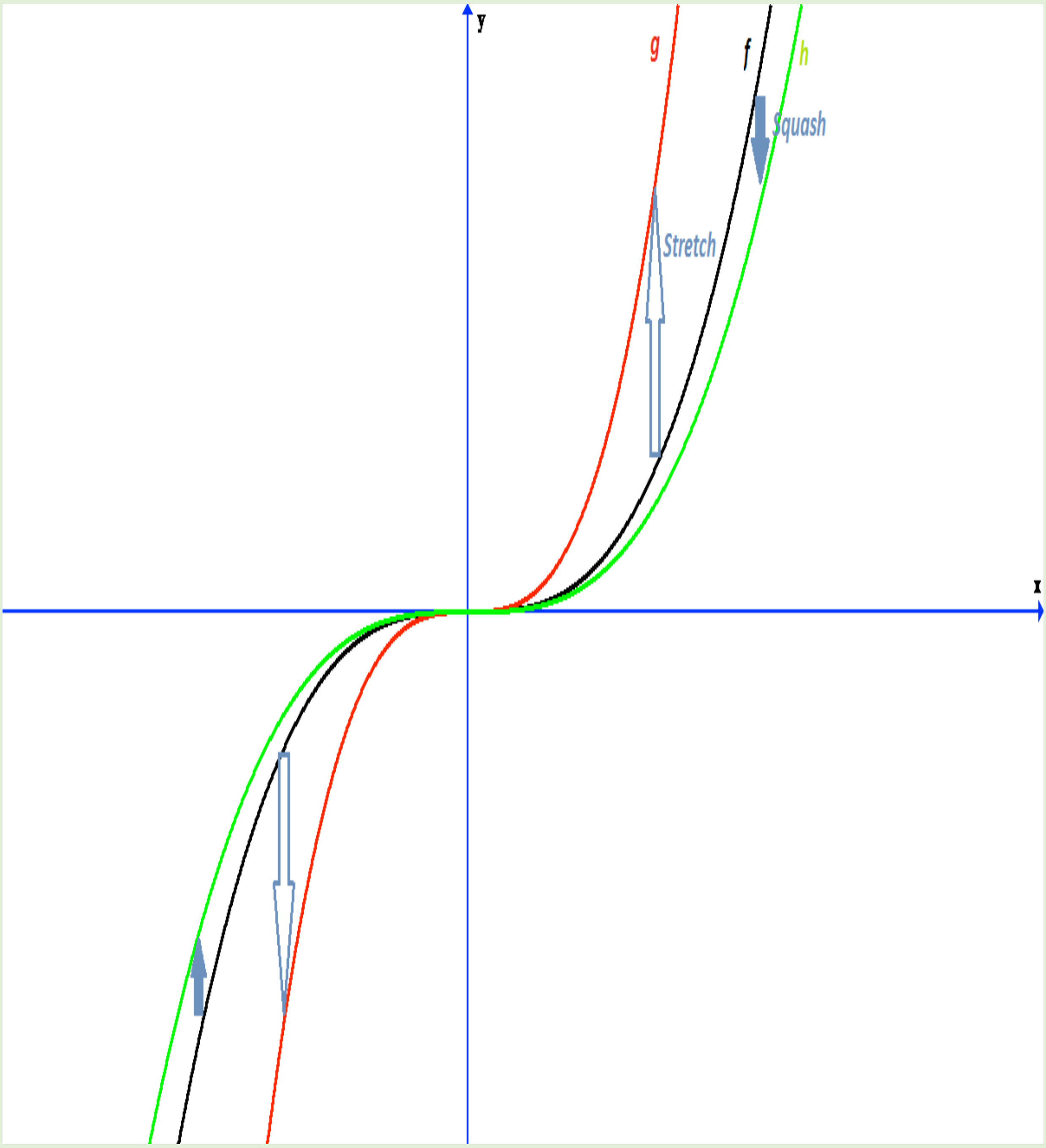
5

a



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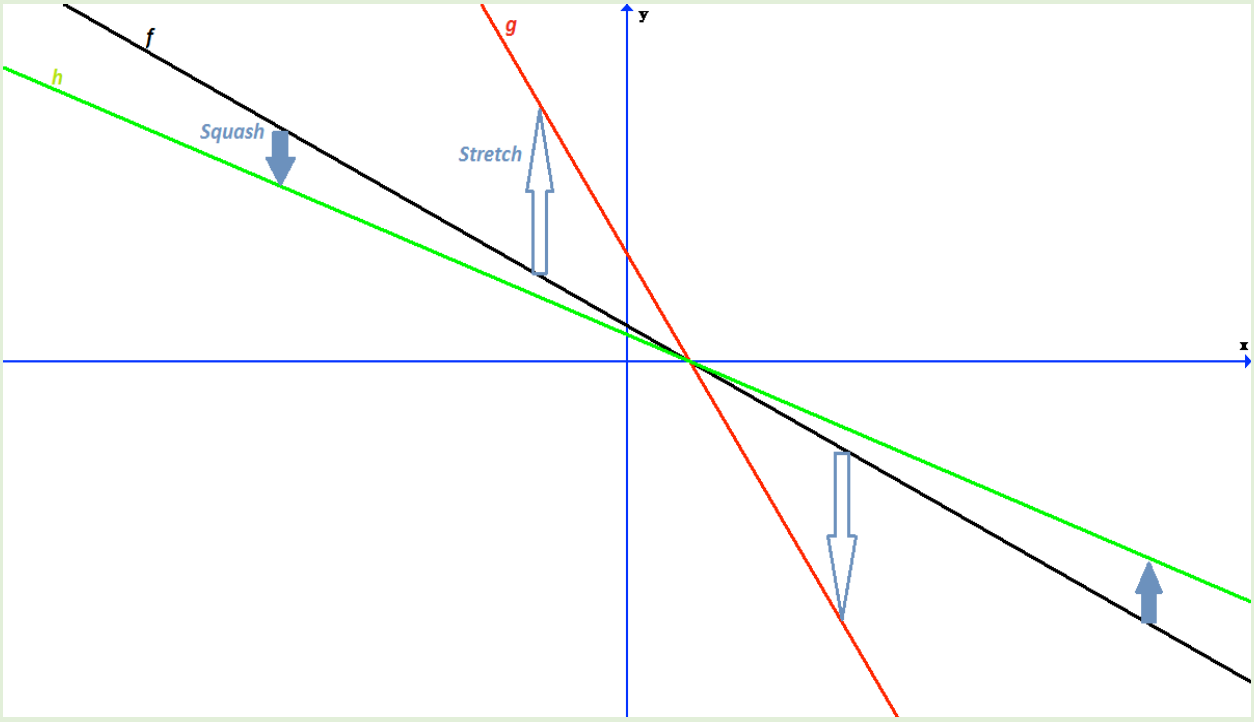
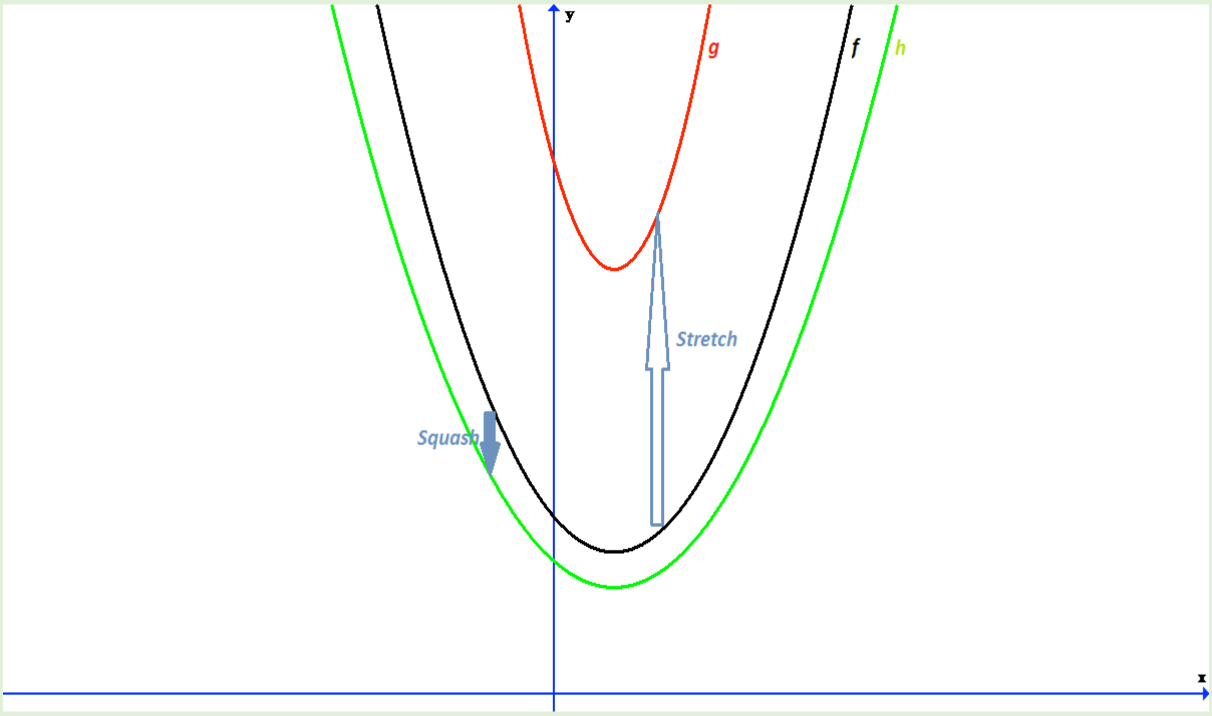
b



c



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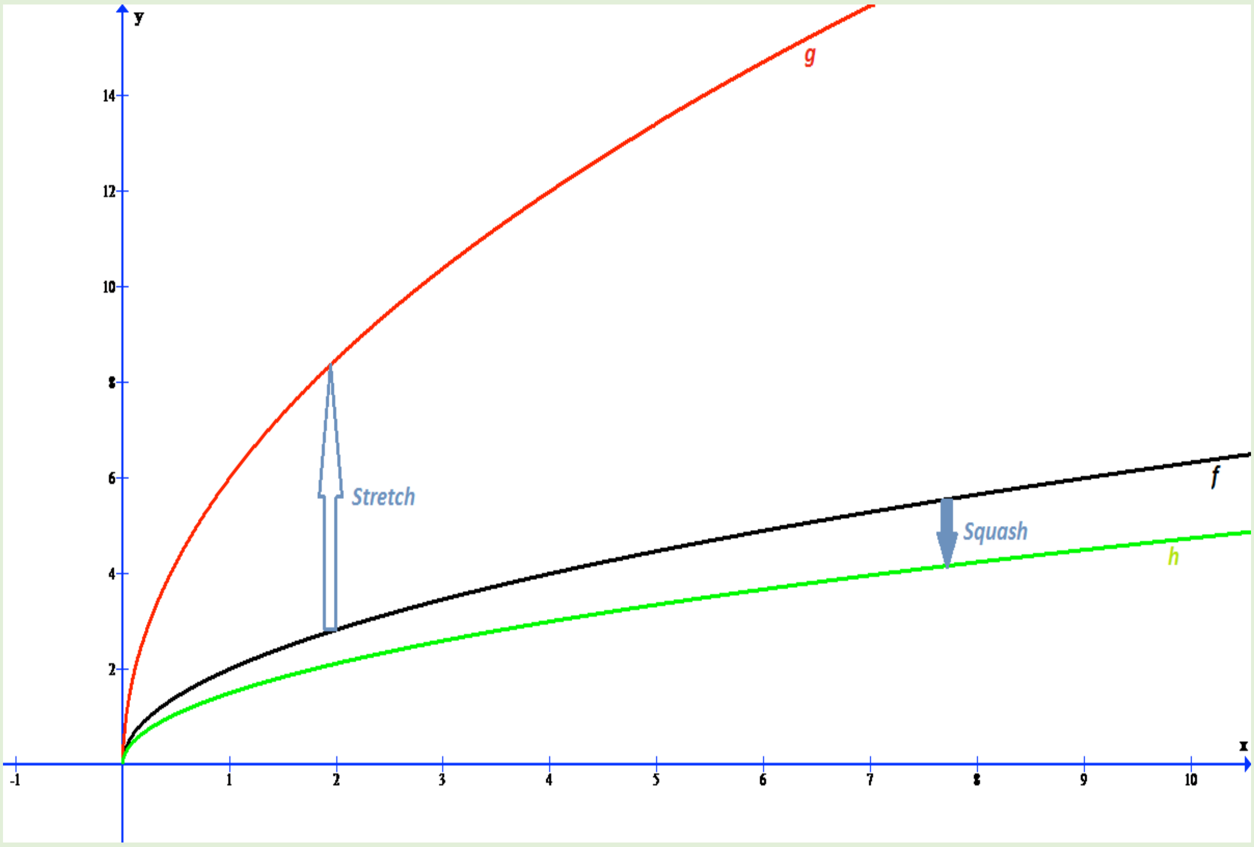
d



Exercise 4 <u>Reflection in the y-axis</u>

1

Points on f	$(2,9)$	$(1,2)$	$(0,1)$	$(-1,0)$	$(-2,-7)$	$(-x,-x^3+1)$
Points on g	$(-2,9)$	$(-1,2)$	$(0,1)$	$(1,0)$	$(2,-7)$	$(x,-x^3+1)$

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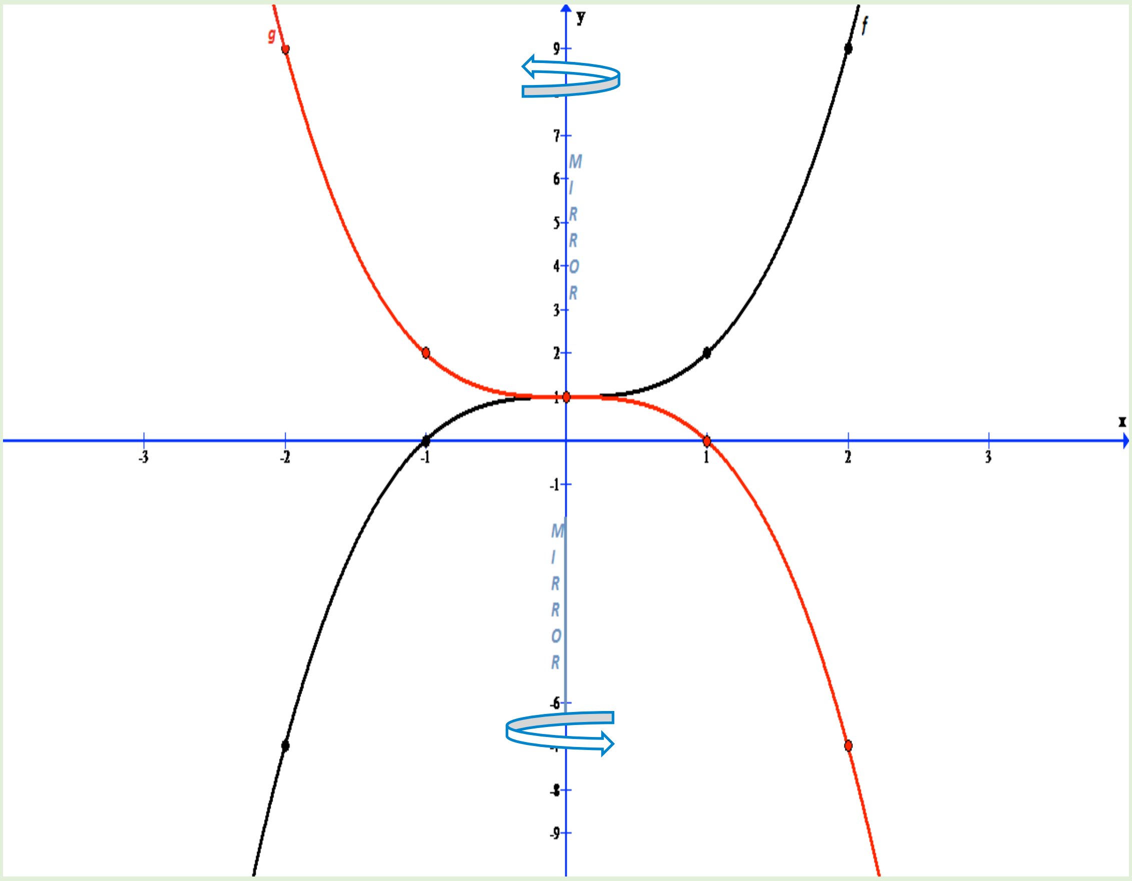
2



Multiplying the x-coordinates by -1 has produced a reflection of f.

g is the mirror image of f with the y-axis acting as the mirror.

3 original function: $f(x) = x^3 + 1$

transformation: reflection in the y-axis

image: $g(x) = -x^3 + 1$

4

	f	g	Are f and g reflections of each other in the y-axis?
a	$f(x) = x + 2$	$g(x) = -x + 2$	YES
b	$f(x) = x^2 + 1$	$g(x) = -x^2 + 1$	NO $g(x) = x^2 + 1$ f is its own reflection in the y-axis!
c	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-x}$	YES

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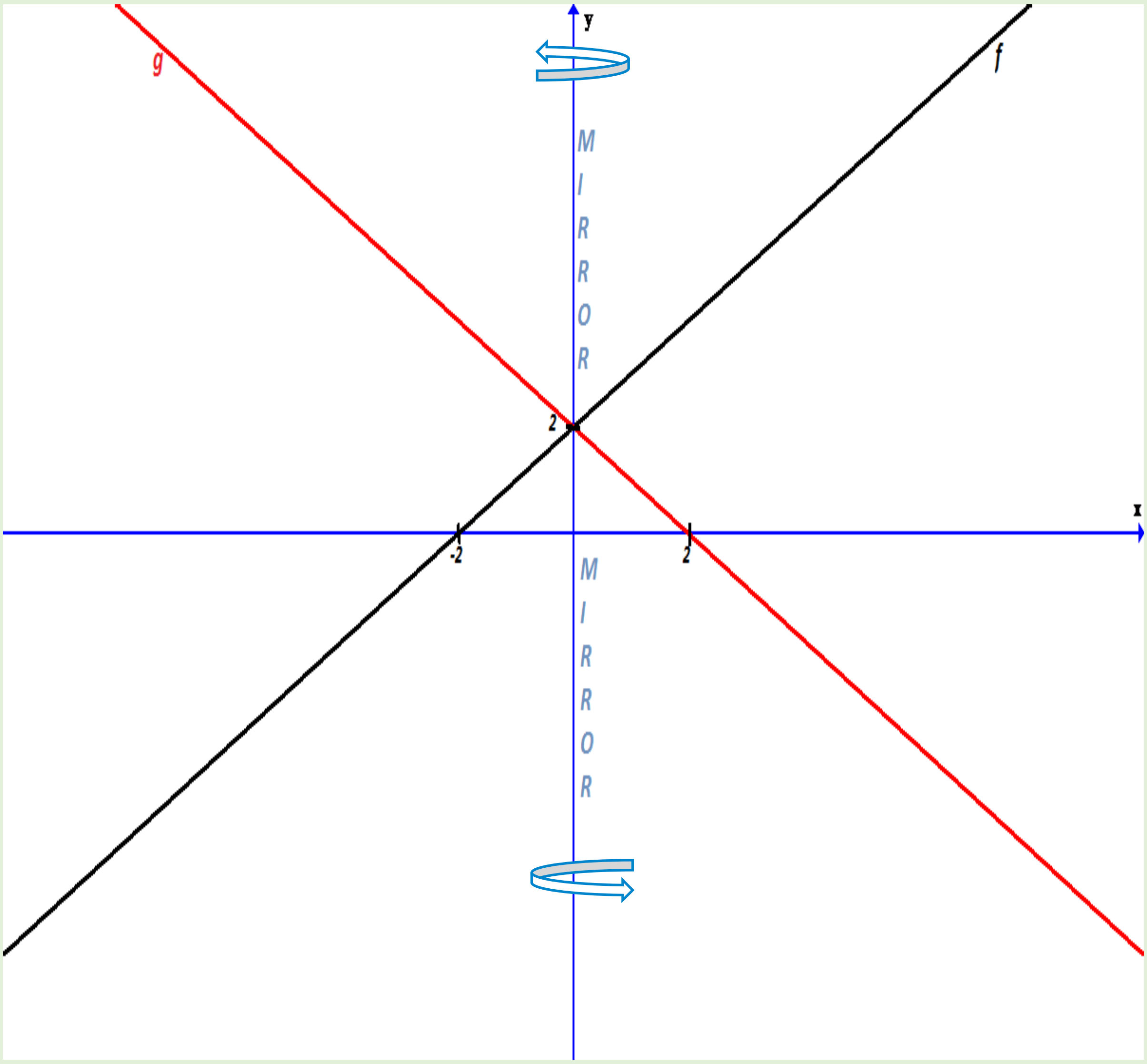
d	$f(x)=x(x+1)^2$	$g(x)=-x(x-1)^2$ Note: $g(x)=-x(-x+1)^2$ $=-x\left[(-x)^2-2x+1\right]$ $=-x\left(x^2-2x+1\right)$ $=-x(x-1)^2$	YES

5

a



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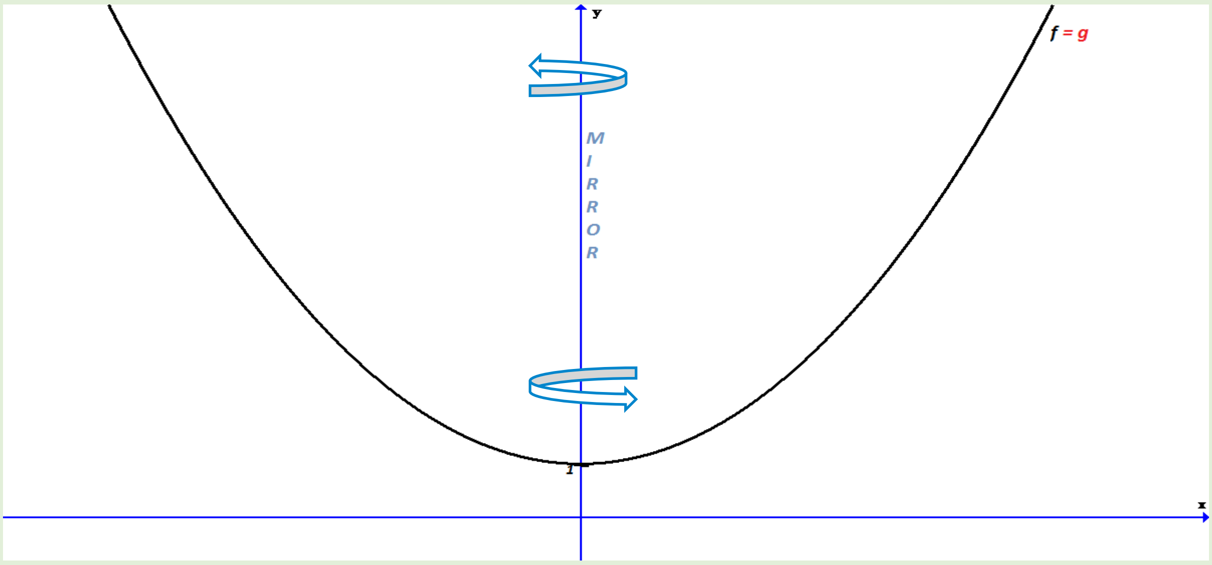
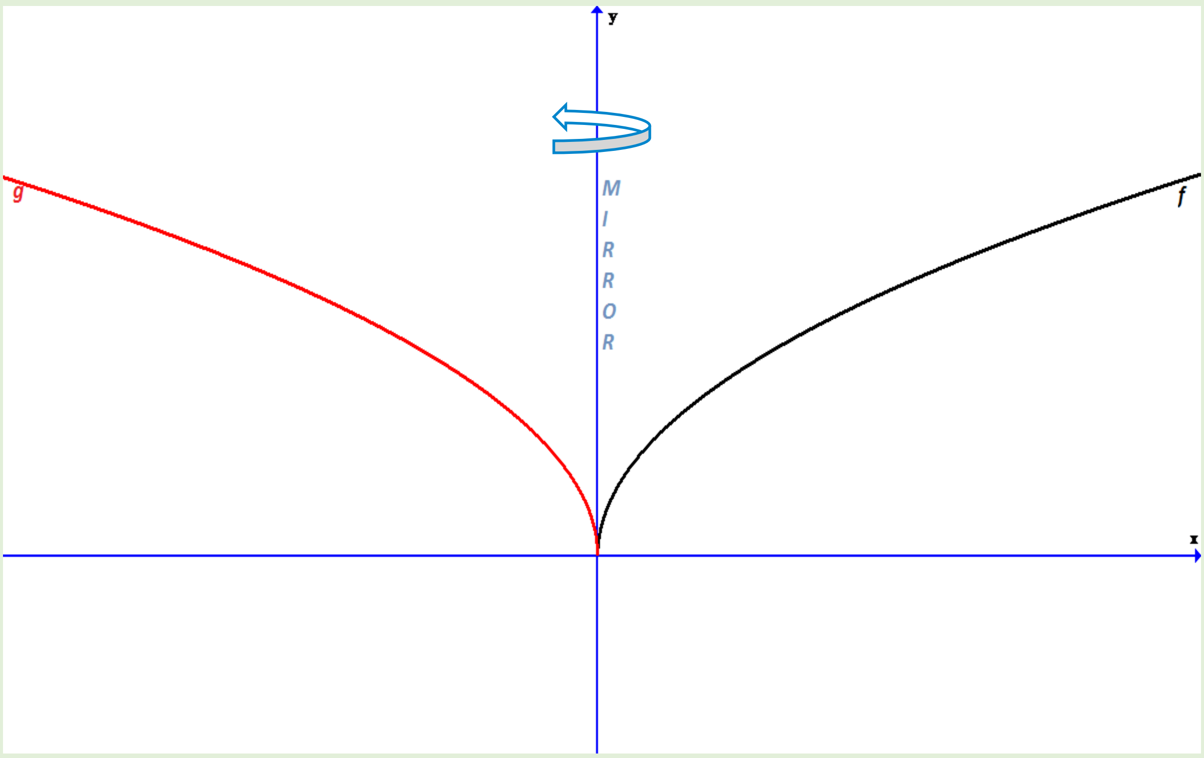
b



c



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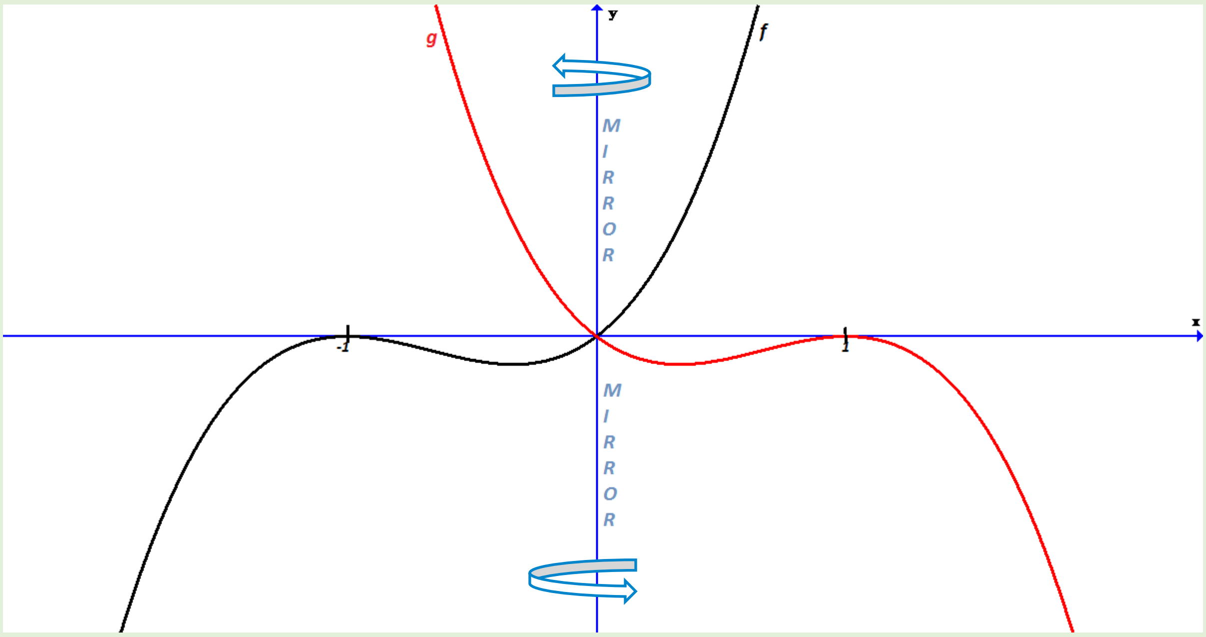
d



Exercise 5 Translations left and right

1a

Points on f	$(-2,3)$	$(-1,0)$	$(0,-1)$	$(1,0)$	$(2,3)$	$\left(x-1,(x-1)^2-1\right)$ or $\left(x-1,x^2-2x\right)$
Points on g	$(-1,3)$	$(0,0)$	$(1,-1)$	$(2,0)$	$(3,3)$	$\left(x,(x-1)^2-1\right)$ or $\left(x,x^2-2x\right)$

b

Points on f	$(-2,3)$	$(-1,0)$	$(0,-1)$	$(1,0)$	$(2,3)$	$\left(x+2,(x+2)^2-1\right)$ or $\left(x+2,x^2+4x+3\right)$
Points on h	$(-4,3)$	$(-3,0)$	$(-2,-1)$	$(-1,0)$	$(0,3)$	$\left(x,(x+2)^2-1\right)$ or $\left(x,x^2+4x+3\right)$

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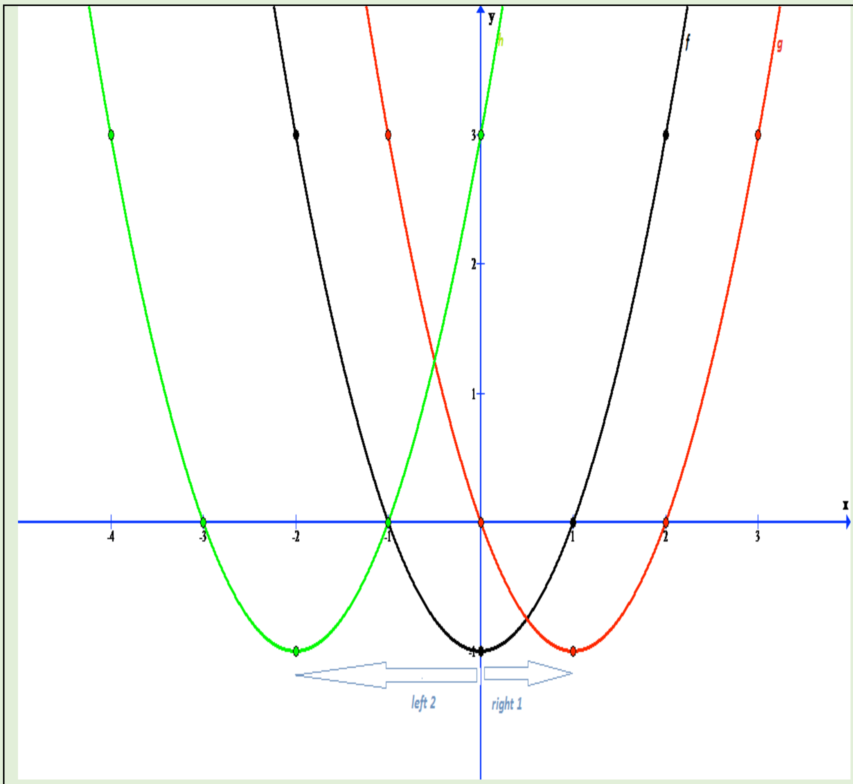
2



Adding 1 to the x-coordinates has shifted f one unit to the right to produce g.

Subtracting 2 from the x-coordinates has shifted *f* two units to the left to produce *h*.

3

a original function: $f(x)=x^2-1$ transformation: translation of 1 unit in the positive direction of the x-axis Image: $g(x)=(x-1)^2-1$ or $g(x)=x^2-2x$	**b** original function: $f(x)=x^2-1$ transformation: translation of 2 units in the negative direction of the x-axis Image: $h(x)=(x+2)^2-1$ or $h(x)=x^2+4x+3$

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4

	f	g	h
a	$f(x)=x^3-1$	$g(x)=(x-2)^3-1$ or $g(x)=x^3-6x^2+12x-9$	$h(x)=(x+e)^3-1$ or $h(x)=x^3+3x^2e+3xe^2+e^3-1$
b	$f(x)=-3x^2+x$	$g(x)=-3(x-2)^2+(x-2)$ $=-3x^2+13x-14$	$h(x)=-3(x+e)^2+(x+e)$ $=-3x^2-6xe-3e^2+x+e$ $=-3x^2+(1-6e)x-3e^2+e$
c	$f(x)=(x+1)^2$	$g(x)=((x-2)+1)^2$ $=(x-1)^2$ or $g(x)=x^2-2x+1$	$h(x)=((x+e)+1)^2$ $=(x+e+1)^2$ or $h(x)=x^2+2(e+1)x+e^2+2e+1$
d	$f(x)=2\sqrt{1-x}$	$g(x)=2\sqrt{1-(x-2)}$ $=2\sqrt{3-x}$	$h(x)=2\sqrt{1-(x+e)}$ $=2\sqrt{1-e-x}$

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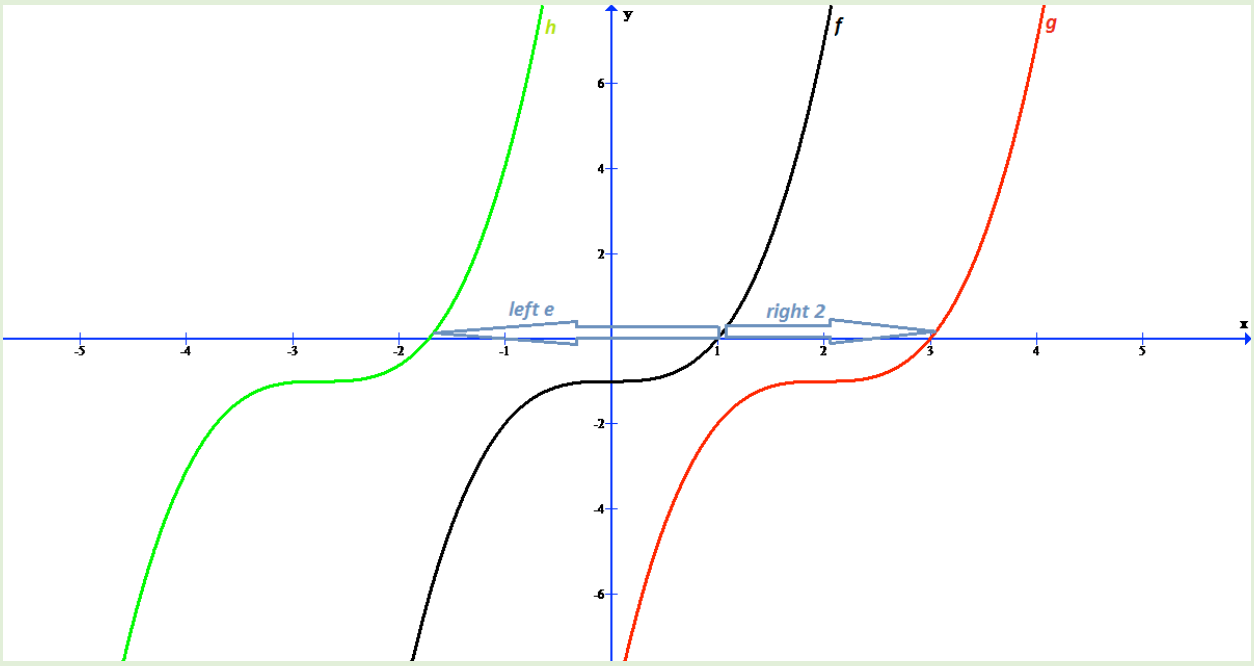
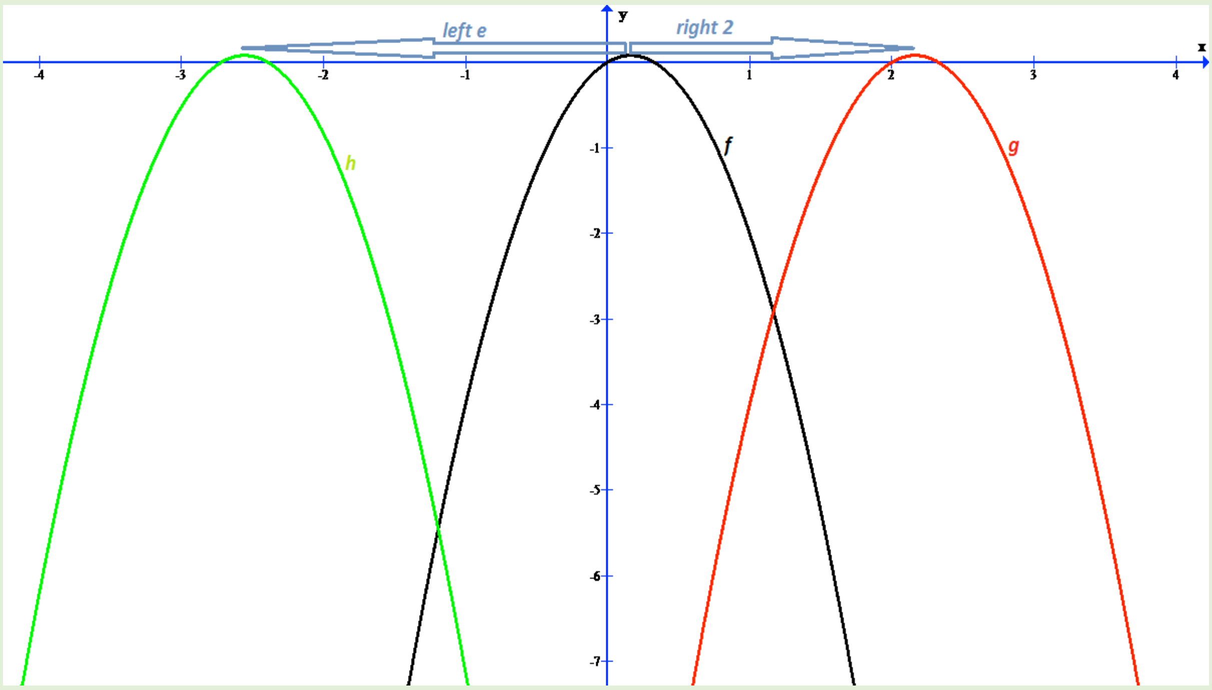
5

a



b



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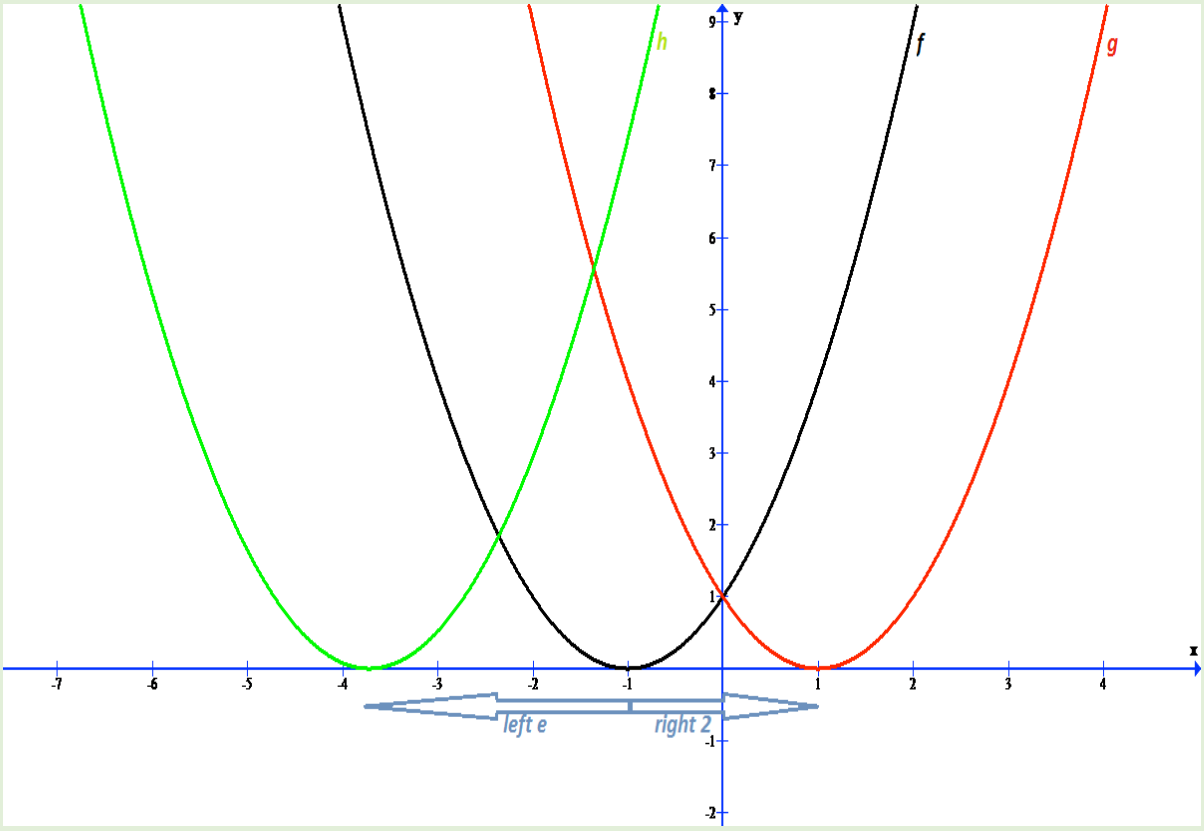
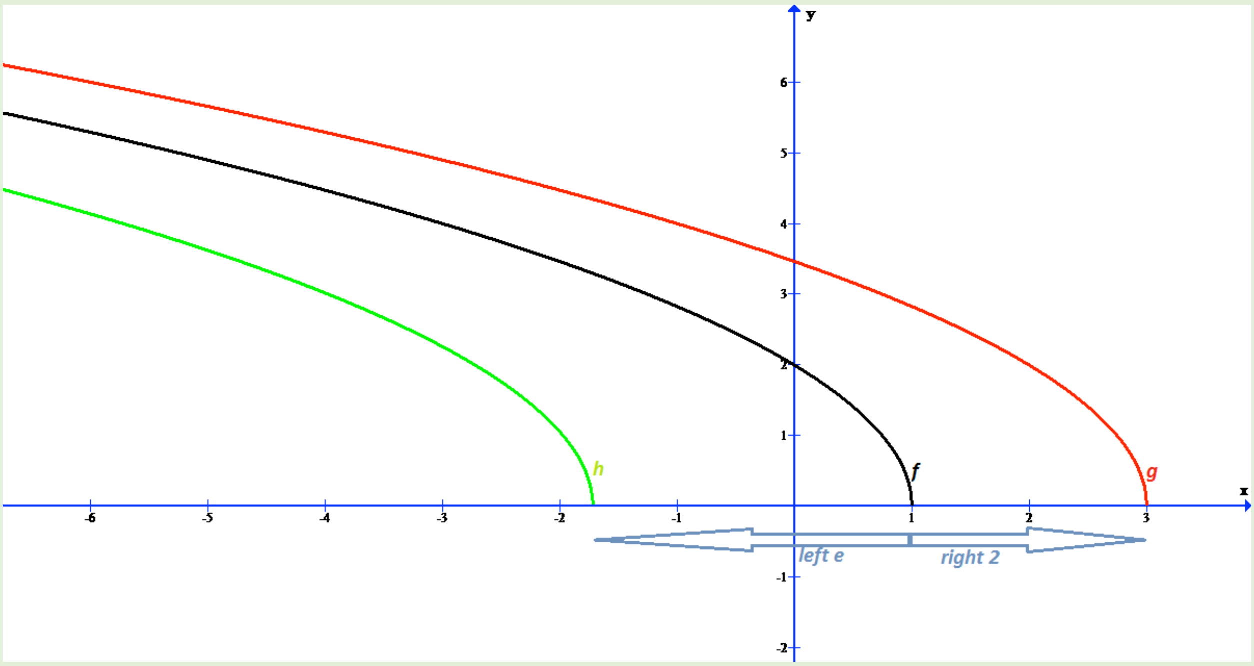
c



d



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Exercise 6 Horizontal dilations

1a

Points on f	$(-2,-8)$	$(-1,-1)$	$(0,0)$	$(1,1)$	$(2,8)$	$\left(\frac{x}{2},\left(\frac{x}{2}\right)^3\right)$ $=\left(\frac{x}{2},\frac{x^3}{8}\right)$
Points on g	$(-4,-8)$	$(-2,-1)$	$(0,0)$	$(2,1)$	$(4,8)$	$\left(x,\left(\frac{x}{2}\right)^3\right)$ $=\left(x,\frac{x^3}{8}\right)$

b

Points on f	$(-2,-8)$	$(-1,-1)$	$(0,0)$	$(1,1)$	$(2,8)$	$\left(2x,(2x)^3\right)$ $=\left(2x,8x^3\right)$
Points on h	$(-1,-8)$	$\left(-\frac{1}{2},-1\right)$	$(0,0)$	$\left(\frac{1}{2},1\right)$	$(1,8)$	$\left(x,(2x)^3\right)$ $=\left(x,8x^3\right)$

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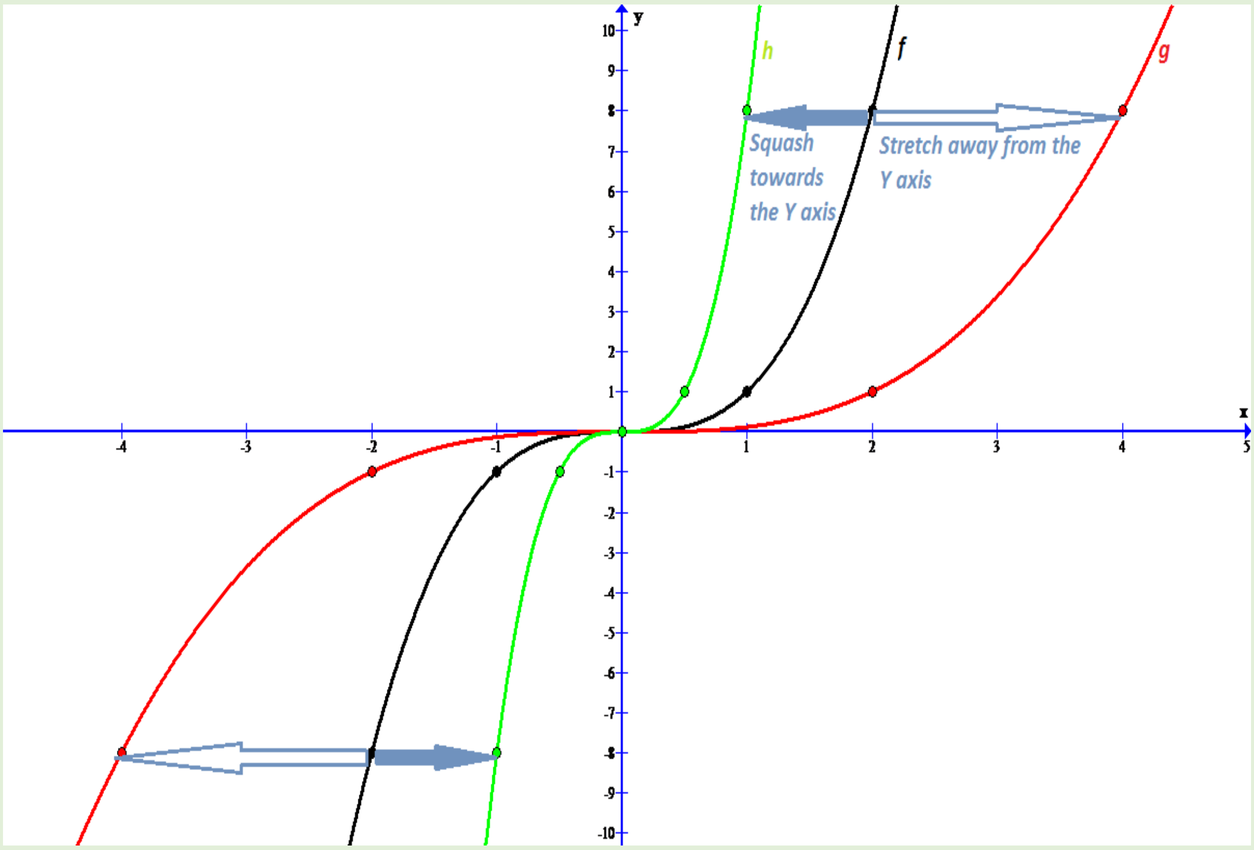
2



3

a original function: $f(x)=x^3$ transformation: dilation from the y-axis of factor 2 Image: $g(x)=\left(\frac{x}{2}\right)^3$ or $g(x)=\frac{x^3}{8}$	**b** original function: $f(x)=x^3$ transformation: dilation parallel to the x-axis of factor $\frac{1}{2}$ Image: $h(x)=(2x)^3$ or $h(x)=8x^3$

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4

	f	g	h
a	$f(x)=-x+1$	$g(x)=-\frac{x}{4}+1$	$h(x)=-\left(\frac{x}{2/3}\right)+1=-\frac{3x}{2}+1$
b	$f(x)=x^3$	$g(x)=\left(\frac{x}{4}\right)^3$ or $g(x)=\frac{x^3}{64}$	$h(x)=\left(\frac{x}{2/3}\right)^3$ $=\left(\frac{3x}{2}\right)^3$ or $h(x)=\frac{27x^3}{8}$
c	$f(x)=(x-1)^2+1$	$g(x)=(\frac{x}{4}-1)^2+1$ $=\left(\frac{1}{4}(x-4)\right)^2+1$ $=\frac{1}{16}(x-4)^2+1$	$h(x)=\left(\frac{x}{2/3}-1\right)^2+1$ $=\left(\frac{3x}{2}-1\right)^2+1$ $=\left(\frac{3}{2}\left(x-\frac{2}{3}\right)\right)^2+1$ $=\frac{9}{4}\left(x-\frac{2}{3}\right)^2+1$
d	$f(x)=2\sqrt{x}$	$g(x)=2\sqrt{\frac{x}{4}}$ $=2\times\frac{\sqrt{x}}{2}$ $=\sqrt{x}$	$h(x)=2\sqrt{\frac{x}{2/3}}$ $=2\sqrt{\frac{3x}{2}}$ $=\sqrt{4\times\frac{3x}{2}}$ $=\sqrt{6x}$

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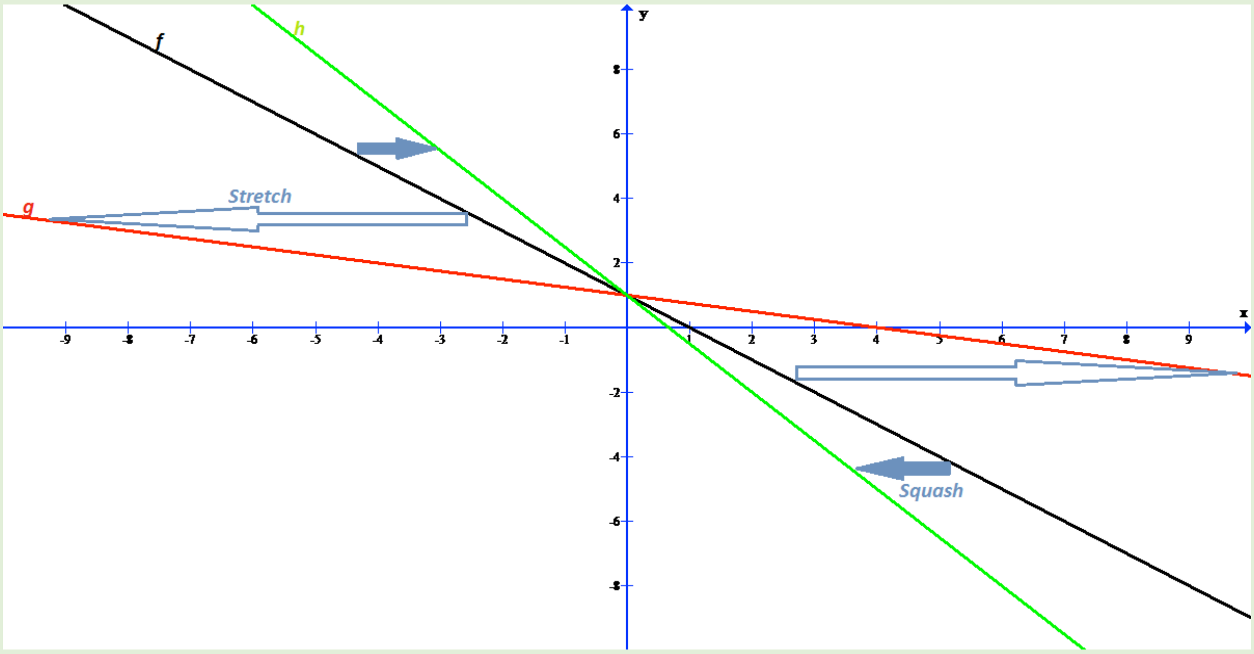
5

a



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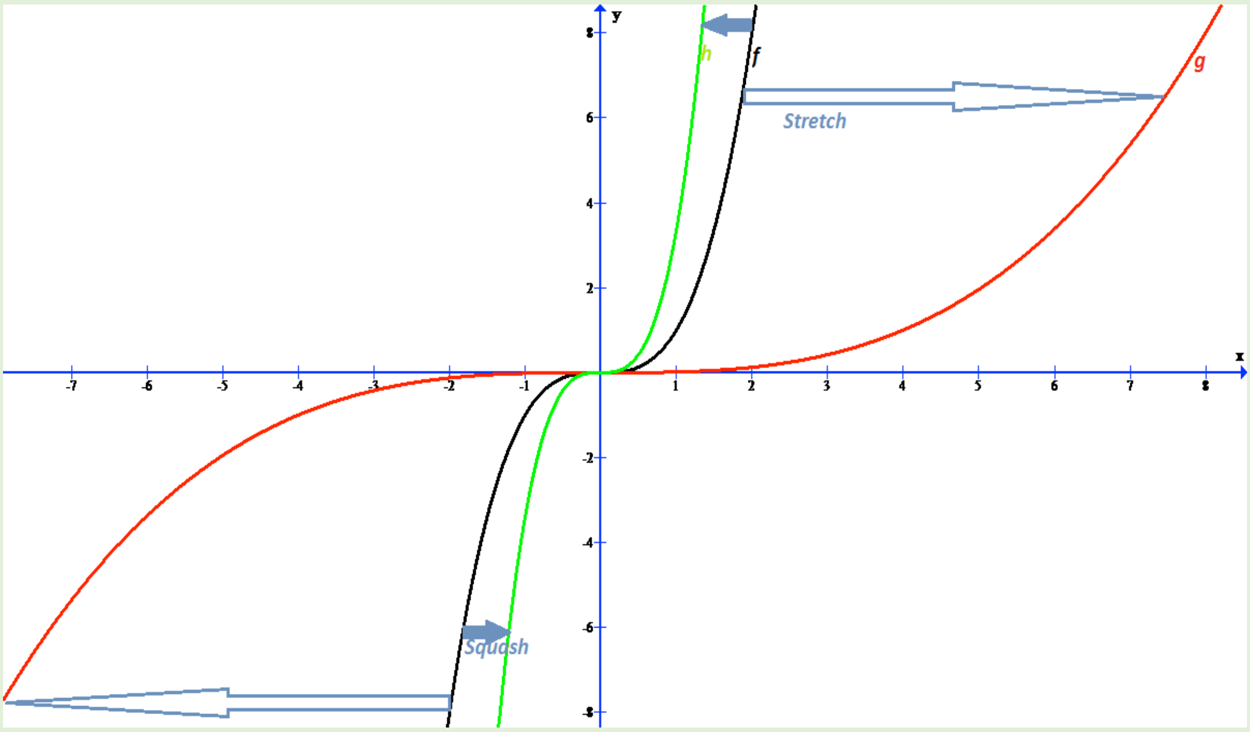

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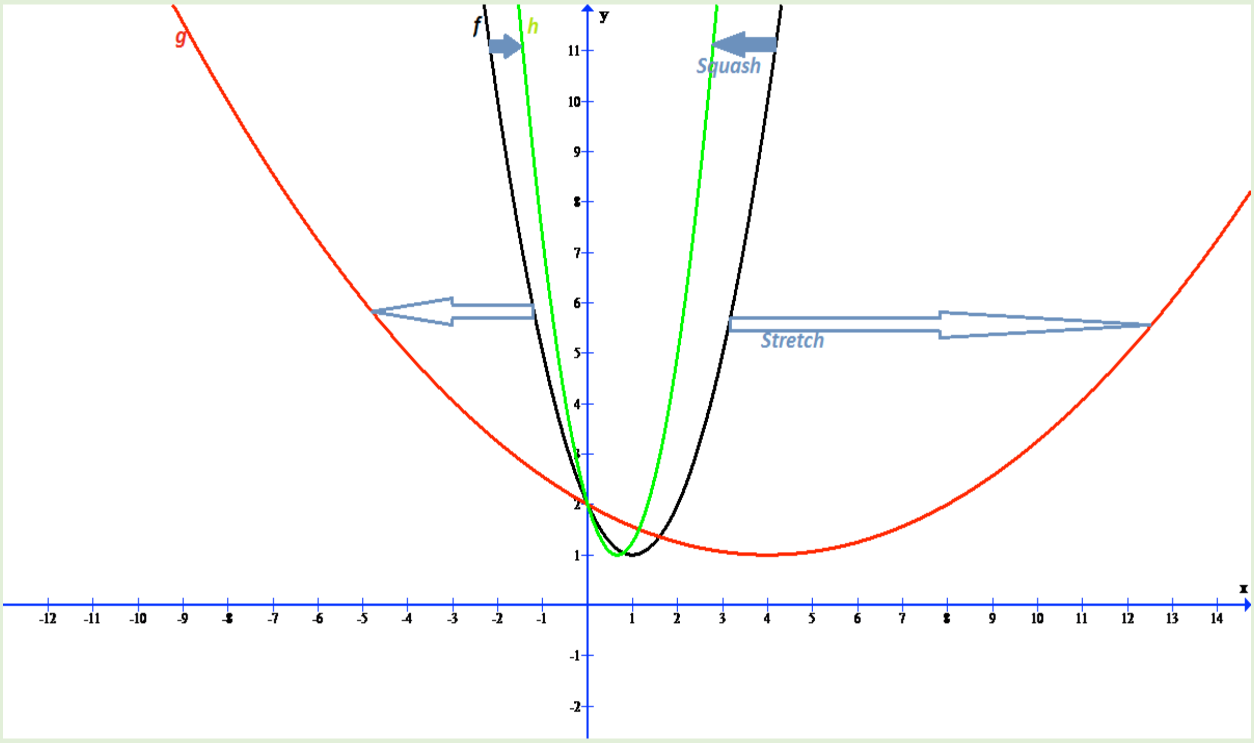
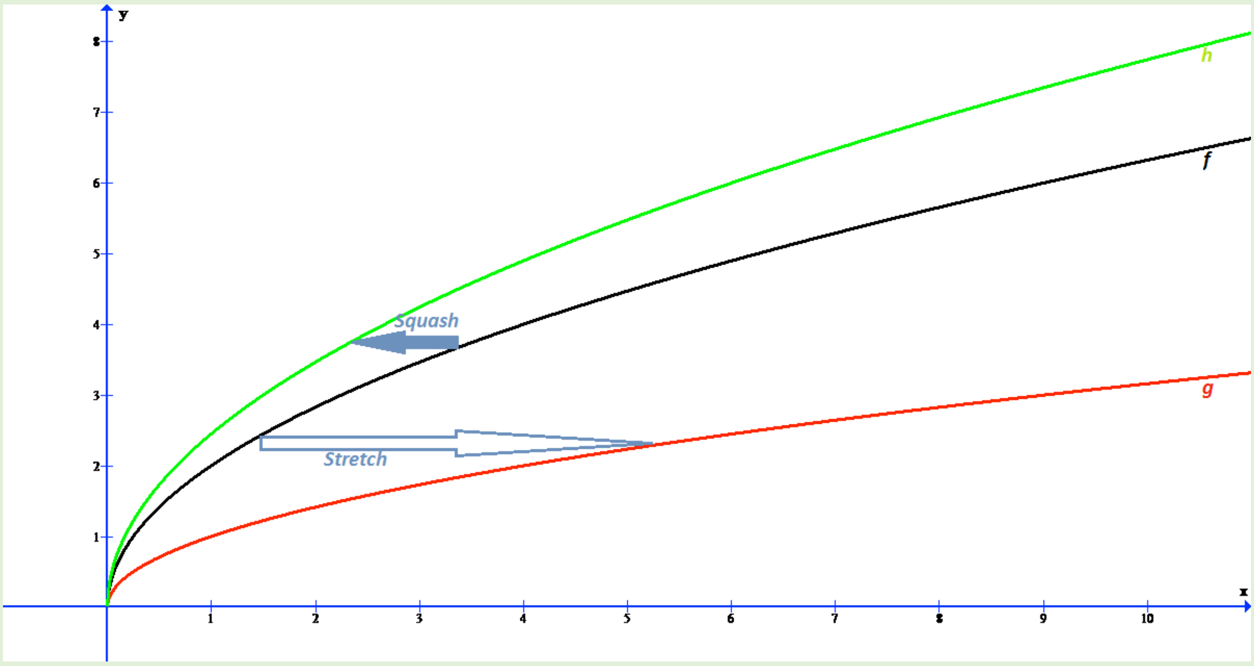
c



d



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APPENDIX A

RELATIONS AND FUNCTIONS

A <u>relation</u> is a rule that tells us about the relationship between certain variables (like x and y).

Given a value for x, the relation tells us how to work out the corresponding value for y.

e.g. 1 $y = x^2 - 1$ is a relation. When x takes the value 0, y takes the value $0^2 - 1 = -1$ When x takes the value 2, y takes the value $2^2 - 1 = 3$ When x takes the value -3, y takes the value $(-3)^2 - 1 = 8$ And so on ...	e.g. 2 $y = \pm\sqrt{x}$ is a relation. When x takes the value 0, y takes the value $\pm\sqrt{0} = 0$ When x takes the value 1, y takes the value $\pm\sqrt{1} = 1 \text{ or } -1$ When x takes the value 4, y takes the value $\pm\sqrt{4} = 2 \text{ or } -2$ And so on…

We can think of a relation as having input (the x values) and output (the y values).

Input *Output*

x ---->---->---- Relation ---->---->---- y

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e.g. 1 continued

Input *Output*

0 ----->--- Relation : $y = x^2 - 1$ -->--- -1

2 ----->--- Relation : $y = x^2 - 1$ ---->--- 3

-3 ---->--- Relation : $y = x^2 - 1$ ---->--- 8

In general,

x ---->--- Relation : $y = x^2 - 1$ -->--- $x^2 - 1$

e.g. 2 continued

Input *Output*

0 ---->------ Relation : $y = \pm\sqrt{x}$ ------>---- 0

1 ----->------- Relation : $y = \pm\sqrt{x}$ ------>----- 1
------>---- -1

4 ---->--- Relation : $y = \pm\sqrt{x}$ ------->---- 2
------>---- -2

In general,

x --->------- Relation : $y = \pm\sqrt{x}$ ----->------ $+\sqrt{x}$
---->----- $-\sqrt{x}$

Note: $+\sqrt{x} = -\sqrt{x}$ only when $x = 0$

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APPENDIX A

A function is a special type of relation.

When each input value generates only one output value, the relation is known as a function.

e.g. 1 continued	e.g. 2 continued
Each input value x , has only one output value y . So $y = x^2 - 1$ is a function.	The input value of 4 generates two output values: 2 and -2. So $y = \pm\sqrt{x}$ is not a function.

When a relation is known to be a function, we can replace "*y*" in the rule with "*f(x)*" or "*g(x)*" or "*h(x)*"….

This is known as "function notation".

e.g. 1 continued	e.g. 2 continued
We can rewrite $y = x^2 - 1$ as $f(x) = x^2 - 1$ or $g(x) = x^2 - 1$ or $h(x) = x^2 - 1$ ……..	We cannot rewrite $y = \pm\sqrt{x}$ as $\cancel{f(x) = \pm\sqrt{x}}$ or $\cancel{g(x) = \pm\sqrt{x}}$ or $\cancel{h(x) = \pm\sqrt{x}}$ ……..

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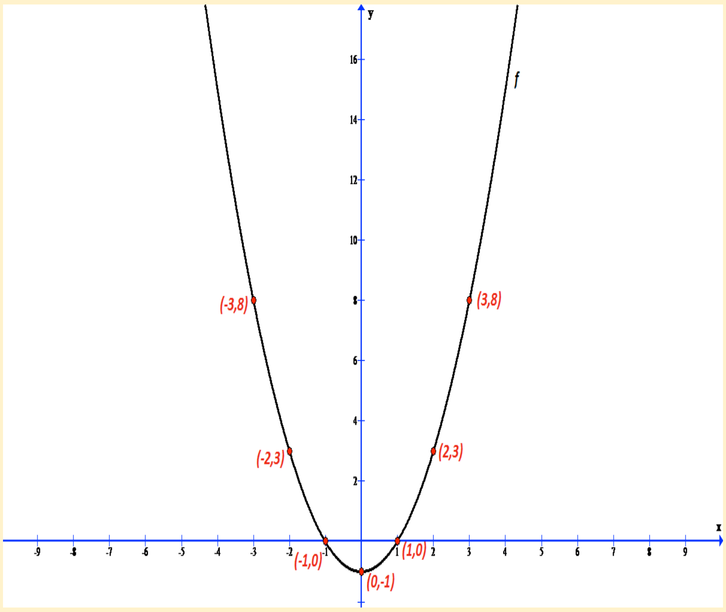
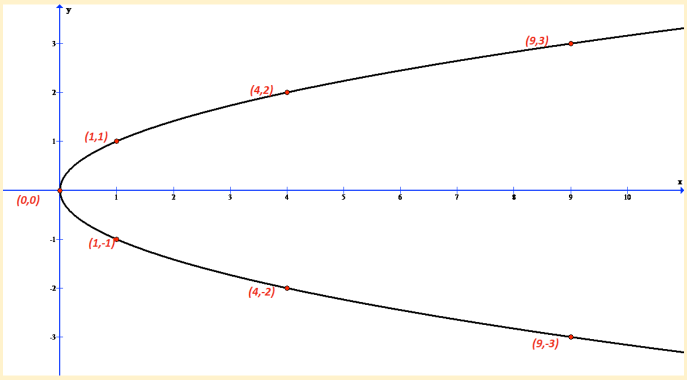
The graph of a function/relation is just a picture of all the coordinate pairs (x, y) that are generated by its rule.

e.g. 1 continued	e.g. 2 continued
$(0,-1)$ will be on the graph of $y = x^2 - 1$. $(2,3)$ will be on the graph of $y = x^2 - 1$. $(-3,8)$ will be on the graph of $y = x^2 - 1$. ………………. All points on the graph of this function *f* will have the form $(x, x^2 - 1)$. The graph of $f(x) = x^2 - 1$ is shown below.  	$(0,0)$ will be on the graph of $y = \pm\sqrt{x}$. $(1,1)$ will be on the graph of $y = \pm\sqrt{x}$. $(1,-1)$ will be on the graph of $y = \pm\sqrt{x}$. $(4,2)$ will be on the graph of $y = \pm\sqrt{x}$. $(4,-2)$ will be on the graph of $y = \pm\sqrt{x}$. ……… All points on the graph of this relation will have the form $(x, \sqrt{x})$ or $(x, -\sqrt{x})$. The graph of $y = \pm\sqrt{x}$ is shown below. 

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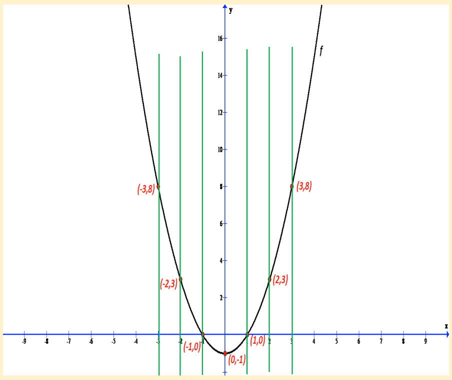
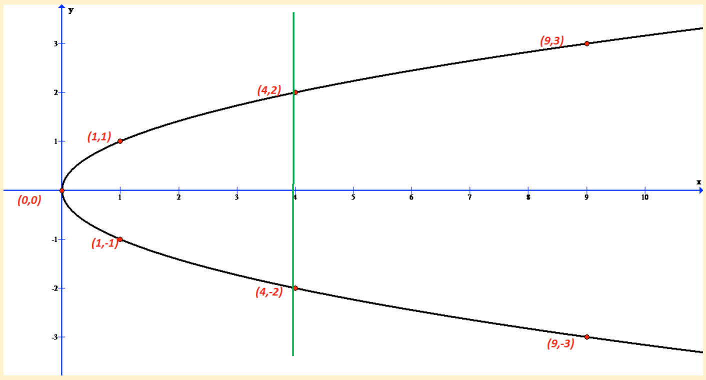
Vertical lines drawn on top of a graph will pick up any input value (x) that has more than one output value (y).

Using this idea we can identify a function by its graph.

Graphs of functions will pass the <u>vertical line test</u> i.e. it is not possible to draw a vertical line anywhere that crosses the graph more than once.

e.g. 1 continued	e.g. 2 continued
The graph of $y = x^2 - 1$ passes the vertical line test, confirming that it is a function.	The graph of $y = \pm\sqrt{x}$ fails the vertical line test, confirming that it is not a function.
	

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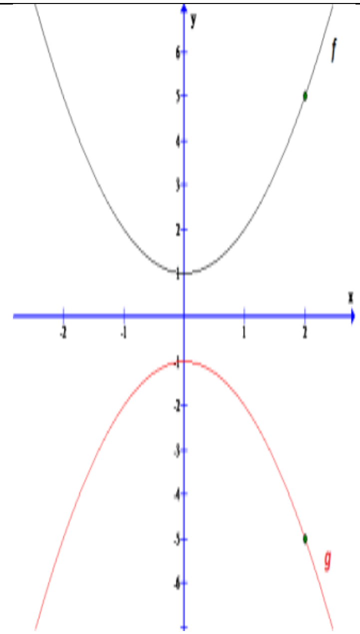
TRANSFORMATIONS SUMMARY

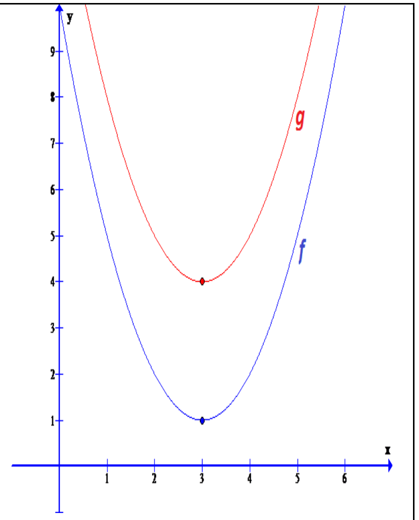
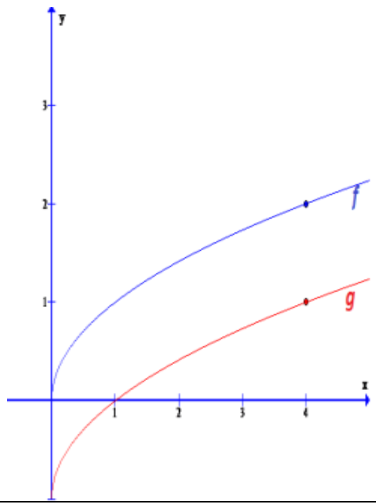
A transformation shifts a point on a plane to a new point.
An entire graph, which is just a collection of points, will be shifted, stretched, squashed or reflected to a new position by a transformation.

Transformation	**Effect of the transformation on a point**	**Effect of the transformation on a rule**	**Visual effect of the transformation on a graph**	**Example**

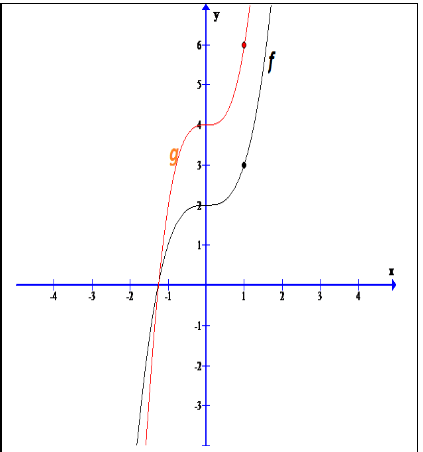
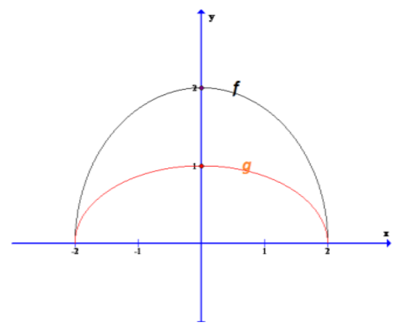
REFLECTION IN THE X-AXIS

reflection in the *x*-axis	The y-value of an ordered pair is multiplied by -1. $(a,b)\to(a,-b)$	The original rule is multiplied by -1 to get the rule of the image. $f(x)\to -f(x)$	The transformation produces a **mirror image** of the original graph with the ***x*-axis acting as a mirror**.	f is mapped to g by a **reflection in the *x*-axis**. point on f: $(2,5)$ image point on g: $(2,-1\times 5)$ $=(2,-5)$ original rule: $f(x)=x^2+1$ image rule: $g(x)=-f(x)$ $=-(x^2+1)$ $=-x^2-1$	

TRANSLATIONS UP AND DOWN

$c>0$ translation of c units in the positive direction of the y-axis	The y-value of an ordered pair has c added to it. $(a,b)\to(a,b+c)$	The original rule has c added to it. $f(x)\to f(x)+c$	The transformation produces a **shift upwards** of the original graph.	f is mapped to g by a translation of **3 units in the positive direction of the y-axis**. point on f: $(3,1)$ original rule: $f(x)=(x-3)^2+1$	 image point on g: $(3,1+3)$ $=(3,4)$ image rule: $g(x)=f(x)+3$ $=(x-3)^2+1+3$ $=(x-3)^2+4$	
$c>0$ translation of c units in the negative direction of the y-axis	The y-value of an ordered pair has c subtracted from it. $(a,b)\to(a,b-c)$	The original rule has c subtracted from it. $f(x)\to f(x)-c$	The transformation produces a **shift downwards** of the original graph.	f is mapped to g by a translation of **1 unit in the negative direction of the y-axis**. point on f: $(4,2)$ original rule: $f(x)=\sqrt{x}$	 image point on g: $(4,2-1)$ $=(4,1)$ image rule: $g(x)=f(x)-1$ $=\sqrt{x}-1$	

DILATION FROM THE X-AXIS

$k>1$ dilation from the x-axis of factor k	The y-value of an ordered pair is multiplied by k. $(a,b)\rightarrow(a,\mathbf{k}b)$	The original rule is multiplied by k. $f(x)\rightarrow \mathbf{k}f(x)$	The transformation produces a **vertical stretch** of the original graph.	f is mapped to g by a **dilation from the x-axis of factor 2.** point on f: $(1,3)$ image point on g: $(1,\mathbf{2}\times 3)$ $=(1,6)$ original rule: $f(x)=x^3+2$ image rule: $g(x)=\mathbf{2}f(x)$ $=\mathbf{2}(x^3+2)$ $=2x^3+4$	
$0<k<1$ dilation from the x-axis of factor k	The y-value of each ordered pair is multiplied by k. $(a,b)\rightarrow(a,\mathbf{k}b)$	The original rule is multiplied by k $f(x)\rightarrow \mathbf{k}f(x)$	The transformation produces a **vertical squash** of the original graph.	f is mapped to g by a **dilation from the x-axis of factor $\frac{1}{2}$** . point on f: $(0,2)$ image point on g: $\left(0,\frac{\mathbf{1}}{\mathbf{2}}\times 2\right)$ $=(0,1)$ original rule: $f(x)=\sqrt{4-x^2}$ image rule: $g(x)=\frac{\mathbf{1}}{\mathbf{2}}f(x)$ $=\frac{\mathbf{1}}{\mathbf{2}}\left(\sqrt{4-x^2}\right)$ $=\frac{1}{2}\sqrt{4-x^2}$	

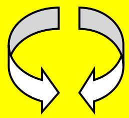
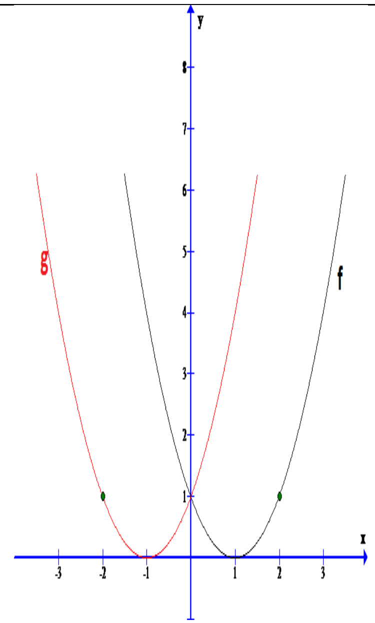
Transformation	**Effect of the transformation on a point**	**Effect of the transformation on a rule**	**Visual effect of the transformation on a graph**	**Example**

REFLECTION IN THE Y-AXIS

<table>
<tr>
<td rowspan="3">reflection in the y-axis</td>
<td rowspan="3">The x-value of an ordered pair is multiplied by -1.

$(a,b)\to(-a,b)$</td>
<td rowspan="3">Every instance of "x" in the original rule is replaced with "$(-x)$".

$f(x)\to f(-x)$</td>
<td rowspan="3">The transformation produces a mirror image of the original graph with the y-axis acting as a mirror.

</td>
<td colspan="2">f is mapped to g by a reflection in the y-axis.</td>
<td rowspan="3">
</td>
</tr>
<tr>
<td>point on f:
$(2,1)$</td>
<td>image point on g:
$(-1\times 2,1)$
$=(-2,1)$</td>
</tr>
<tr>
<td>original rule:

$f(x)=x^2-2x+1$</td>
<td>image rule:

$g(x)=f(-x)$
$=(-x)^2-2(-x)+1$
$=x^2+2x+1$</td>
</tr>
</table>

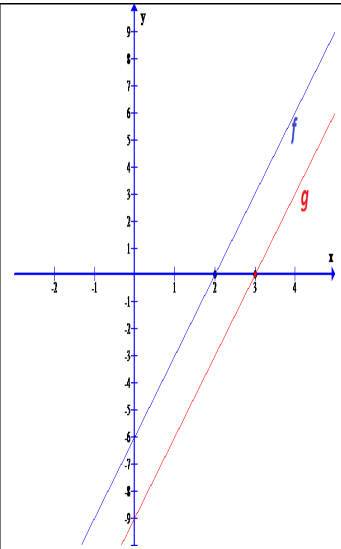
TRANSLATIONS LEFT AND RIGHT

<table>
<tr>
<td rowspan="3">$c>0$

translation of c units in the positive direction of the x-axis</td>
<td rowspan="3">The x-value of an ordered pair has c added to it.

$(a,b)\to(a+\mathbf{c},b)$</td>
<td rowspan="3">Every instance of “x” in the original rule is replaced with “$(x-c)$”.

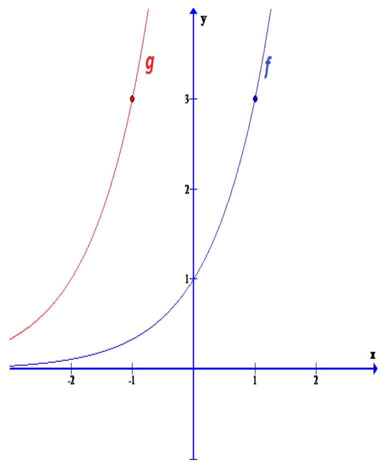
$f(x)\to f(\mathbf{x-c})$</td>
<td rowspan="3">The transformation produces a shift to the right of the original graph.

⇨</td>
<td colspan="2">f is mapped to g by a translation of 1 unit in the positive direction of the x-axis.</td>
<td rowspan="3"></td>
</tr>
<tr>
<td>point on f:
$(2,0)$</td>
<td>image point on g:
$(2+\mathbf{1},0)$
$=(3,0)$</td>
</tr>
<tr>
<td>original rule:
$f(x)=3x-6$</td>
<td>image rule:
$g(x)=f(\mathbf{x-1})$
$=3(\mathbf{x-1})-6$
$=3x-3-6$
$=3x-9$</td>
</tr>
<tr>
<td rowspan="3">$c>0$

translation of c units in the negative direction of the x-axis</td>
<td rowspan="3">The x-value of an ordered pair has c subtracted from it.

$(a,b)\to(a-\mathbf{c},b)$</td>
<td rowspan="3">Every instance of “x” in the original rule is replaced with “$(x+c)$”.

$f(x)\to f(\mathbf{x+c})$</td>
<td rowspan="3">The transformation produces a shift to the left of the original graph.

⇦</td>
<td colspan="2">f is mapped to g by a translation of 2 units in the negative direction of the x-axis.</td>
<td rowspan="3"></td>
</tr>
<tr>
<td>point on f:
$(1,3)$</td>
<td>image point on g:
$(1-\mathbf{2},3)$
$=(-1,3)$</td>
</tr>
<tr>
<td>original rule:
$f(x)=3^x$</td>
<td>image rule:
$g(x)=f(\mathbf{x+2})$
$=3^{(x+2)}$
$=3^{x+2}$</td>
</tr>
</table>

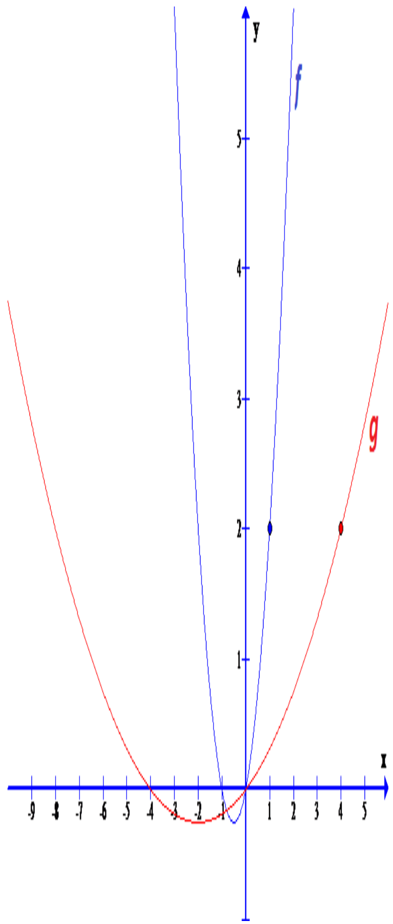
DILATION FROM THE Y-AXIS

<table>
<tr>
<td rowspan="3">$k>1$

dilation from the y-axis of factor k</td>
<td rowspan="3">The x-value of an ordered pair is multiplied by k.

$(a,b)\rightarrow(ka,b)$</td>
<td rowspan="3">Every instance of "x" in the original rule is replaced with "$\left(\frac{1}{k}x\right)$" or "$\left(\frac{x}{k}\right)$".

$f(x)\rightarrow f\left(\frac{x}{k}\right)$</td>
<td rowspan="3">The transformation produces a horizontal stretch of the original graph.

⟺</td>
<td colspan="2">f is mapped to g by a dilation from the y-axis of factor 4.</td>
<td rowspan="3">
</td>
</tr>
<tr>
<td>point on f:

$(1,2)$</td>
<td>image point on g:

$(4\times 1,2)$
$=(4,2)$</td>
</tr>
<tr>
<td>original rule:

$f(x)=x^2+x$</td>
<td>image rule:

$g(x)=f\left(\frac{x}{4}\right)$
$=\left(\frac{x}{4}\right)^2+\left(\frac{x}{4}\right)$
$=\frac{x^2}{16}+\frac{x}{4}$

or

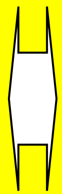
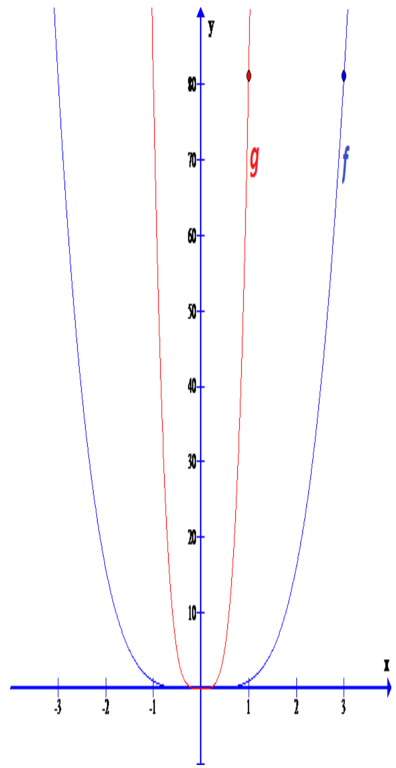
$g(x)=\frac{1}{16}x^2+\frac{1}{4}x$</td>
</tr>
</table>

<table>
<tr>
<td rowspan="3">0<<i>k</i><1

dilation from the <i>y</i>-axis of factor <i>k</i></td>
<td rowspan="3">The x-value of an ordered pair is multiplied by k.

$(a,b)\to(\mathbf{k}a,b)$</td>
<td rowspan="3">Every instance of "x" in the original rule is replaced with "$\left(\frac{1}{k}x\right)$" or "$\left(\frac{x}{k}\right)$".

$f(x)\to f\left(\frac{\mathbf{x}}{\mathbf{k}}\right)$</td>
<td rowspan="3">The transformation produces a horizontal squash of the original graph.

</td>
<td colspan="2">f is mapped to g by a dilation from the y-axis of factor $\frac{1}{3}$.</td>
<td rowspan="3"></td>
</tr>
<tr>
<td>point on f:

$(3,81)$</td>
<td>image point on g:
$\left(\frac{\mathbf{1}}{\mathbf{3}}\times 3,81\right)$
$=(1,81)$</td>
</tr>
<tr>
<td>original rule:

$f(x)=x^4$</td>
<td>image rule:

$g(x)=f\left(\frac{\mathbf{x}}{\mathbf{1/3}}\right)$
$=f(\mathbf{3x})$
$=(\mathbf{3x})^4$
$=81x^4$</td>
</tr>
</table>

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